

On signless Laplacian coefficients of unicyclic graphs with given matching number *

Jie Zhang, Xiao-Dong Zhang[†]

Department of Mathematics, Shanghai Jiao Tong University

800 DongChuan road, Shanghai, 200240, P.R. China

December 21, 2012

Abstract

Let G be an unicyclic graph of order n and let $Q_G(x) = \det(xI - Q(G)) = \sum_{i=1}^n (-1)^i \varphi_i x^{n-i}$ be the characteristic polynomial of the signless Laplacian matrix of a graph G . We give some transformations of G which decrease all signless Laplacian coefficients in the set $\mathcal{G}(n, m)$. $\mathcal{G}(n, m)$ denotes all n -vertex unicyclic graphs with matching number m . We characterize the graphs which minimize all the signless Laplacian coefficients in the set $\mathcal{G}(n, m)$ with odd (resp. even) girth. Moreover, we find the extremal graphs which have minimal signless Laplacian coefficients in the set $\mathcal{G}(n)$ of all n -vertex unicyclic graphs with odd (resp. even) girth.

Key words: Signless Laplacian coefficients; TU-subgraph; Matching; Unicyclic graph

AMS Classifications: 05C50, 05C07.

1 Introduction

Let G be a simple undirect unicyclic graph. $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For every unicyclic graph G , $|V(G)| = |E(G)|$. Let

*This work is supported by National Natural Science Foundation of China (No:10971137), the National Basic Research Program (973) of China (No.2006CB805900), and a grant of Science and Technology Commission of Shanghai Municipality (STCSM, No: 09XD1402500).

[†]Corresponding author (*E-mail address:* xiaodong@sjtu.edu.cn)

$d(v_i)$ denote the degree of vertex v_i , and let $D(G) = \text{diag}(d(v_1), d(v_2), \dots, d(v_n))$ be the diagonal matrix of G . Furthermore, let $A(G)$ be the adjacent matrix of G . The Laplacian matrix of G is $L(G) = D(G) - A(G)$, and the Laplacian characteristic polynomial is denoted by $L_G(x) = \det(xI - L(G)) = \sum_{k=1}^n (-1)^k c_k x^{n-k}$. The Laplacian coefficients $c_k(G)$ of a graph G can be expressed in terms of subtree structures of G by the following result of Kelmans and Chelnokov [16]. Let F be a spanning forest of G with k components T_1, T_2, \dots, T_s , T_i has $|V(T_i)|$ vertices, let

$$\gamma(F) = \prod_{i=1}^k |V(T_i)|,$$

Theorem 1.1 ([16]) *Let \mathcal{F}_k be the set of all spanning forests of G with exactly k components. Then the Laplacian coefficient $c_{n-k}(G)$ is expressed by $c_{n-k}(G) = \sum_{F \in \mathcal{F}_k} \gamma(F)$.*

Recently, the study on the Laplacian coefficients have attracted much attention. Mohar [17] first investigate the Laplacian coefficients of acyclic graphs under the partial order \preceq . Zhang et al. [21] investigated ordering trees with diameters 3 and 4 by the Laplacian coefficients. Ilić [13] determined the n -vertex tree of fixed diameter which minimizes the Laplacian coefficients. Ilić [14] determined the n -vertex tree with given matching number having the minimum Laplacian coefficients. He and Li [11] studied the ordering of all n -vertex trees with a perfect matching by Laplacian coefficients. Ilić and Ilić [12] studied the n -vertex trees with fixed pendent vertex number and 2-degree vertex number which have minimum Laplacian coefficients. Stevanović and Ilić [19] investigated the Laplacian coefficients of unicyclic graphs. Tan [20] characterized the determined the n -vertex unicyclic graph with given matching number which minimizes all Laplacian coefficients. He and Shan [10] studied the Laplacian coefficients of bicyclic graphs.

The signless Laplacian matrix of G , $Q(G) = D(G) + A(G)$, which is related to $L(G)$, has also been studied recently (see [1-5, [18]]). The signless Laplacian characteristic polynomial is denoted by $Q_G(x) = \det(xI - Q(G)) = \sum_{i=1}^n (-1)^i \varphi_i x^{n-i}$. Using the notation from [2], [18], a TU-subgraph of G is the spanning subgraph of G whose components are trees or odd unicyclic graphs. Assume that a TU-subgraph H of G contains c odd unicyclic graphs and s trees T_1, \dots, T_s . The weight of H can be expressed by $W(H) = 4^c \prod_{i=1}^s n_i$, in which n_i is the number of T_i . If H contains no tree, let $W(H) = 4^c$. If H is empty, in other words, H does not exist, let $W(H) = 0$. The signless Laplacian coefficients $\varphi_i(G)$ can be expressed in terms of the weight of TU-subgraphs of G .

Theorem 1.2 ([2],[18]) *Let G be a connected graph. For φ_i as above, we have $\varphi_0 = 1$ and*

$$\varphi_i = \sum_{H_i} W(H_i), i = 1, \dots, n;$$

where the summation runs over all TU-subgraphs H_i of G with i edges.

From Theorem 1.2, it is obvious that for a n -vertex connected unicyclic graph G , $\varphi_1(G) = 2n$. Let g be the girth of G , which is the length of the cycle. if g is odd, $\varphi_n(G) = 4$. When g is even, G is bipartite graph, then $\varphi_n(G) = 0$, and $\varphi_{n-1}(G)$ counts the number of all spanning trees of G , thus $\varphi_{n-1}(G) = n \cdot g$. When G is bipartite graph, $L(G)$ and $Q(G)$ have the same characteristic polynomial, so $c_i(G) = \varphi_i(G), i = 0, 1, 2, \dots, n$, and the expression of φ_i in Theorem 1.2 is equivalence to the expression of c_i in Theorem 1.1.

The eigenvalues of $L(G)$ and $Q(G)$ are denoted by $\mu_1(G) \geq \dots \geq \mu_n(G) = 0$ and $\nu_1(G) \geq \dots \geq \nu_n(G)$, respectively. The incidence energy of G , $IE(G)$ for short, is defined as $IE(G) = \sum_{i=1}^n \sqrt{\nu_i(G)}$ (see [7],[8],[15]).

Mirzakhah and Kiani [18] presented a connection between the incidence energy and the signless Laplacian coefficients.

Theorem 1.3 ([18]) *Let G and G' be two graphs of order n . If $\varphi_i(G) \leq \varphi_i(G')$ for $1 \leq i \leq n$, then $IE(G) \leq IE(G')$ and $IE(G) < IE(G')$ if $\varphi_i(G) < \varphi_i(G')$ for some i holds.*

Let G be a graph which is not a star, let v be a vertex with degree $p+1$ in G , such that it is adjacent with $\{u, v_1, v_2, \dots, v_p\}$, where $\{v_1, v_2, \dots, v_p\}$ are pendent vertices. The graph $G' = \sigma(G, v)$ is obtained from deleting edges vv_1, vv_2, \dots, vv_p and adding edges uv_1, uv_2, \dots, uv_p .

Theorem 1.4 ([18]) *Let G be a connected graph and $G' = \sigma(G, v)$, then $\varphi_i(G) \geq \varphi_i(G')$, for every $0 \leq i \leq n$, with equality if and only if either $i \in \{0, 1, n\}$ when G is non-bipartite, or $i \in \{0, 1, n-1, n\}$ otherwise.*

Let $G = G_1|u : G_2|v$ be the graph obtained from two disjoint graphs G_1 and G_2 by joining a vertex u of G_1 and a vertex v of G_2 by an edge. For any graph G and $v \in V(G)$, let $L_{G|v}(x)$ be the principal submatrix of $L_G(x)$ obtained by deleting the row and column corresponding to the vertex v .

Theorem 1.5 ([6]) *If $G = G_1|u : G_2|v$, then $L_G(x) = L_{G_1}(x)L_{G_2}(x) - L_{G_1}(x)L_{G_2|v}(x) - L_{G_2}(x)L_{G_1|u}(x)$.*

Theorem 1.6 ([9]) *If G be a connected graph with n vertices which consists of a subgraph H ($V(H) \geq 2$) and $n - |V(H)|$ pendent vertices attached to a vertex v in H , then $L_G(x) = (x - 1)^{(n - |V(H)|)} L_H(x) - (n - |V(H)|) x (x - 1)^{(n - |V(H)| - 1)} L_{H|v}(x)$.*

Throughout this paper, we use the following notations. Let $\mathcal{G}(n)$ be the set of all unicyclic graphs of order n . Let $\mathcal{G}(n, m)$ be the set of all n -vertex unicyclic graphs with matching number m . Let $\mathcal{G}(n, m) = \mathcal{G}_{g_1}(n, m) \cup \mathcal{G}_{g_2}(n, m)$, where $\mathcal{G}_{g_1}(n, m)$ (resp. $\mathcal{G}_{g_2}(n, m)$) denotes the subset of $\mathcal{G}(n, m)$ with odd (resp. even) girth. Similarly, we can define the subsets $\mathcal{G}_{g_1}(n)$ and $\mathcal{G}_{g_2}(n)$ of $\mathcal{G}(n)$.

Denote $M(G)$ a maximum matching of G , for $G \in \mathcal{G}(n, m)$, $|M(G)| = m$. Using notations from [20], for a nonpendent edge uv of G , $E_{uv}^u = E_{vu}^u$ denote the set of all edges incident to u except uv . u is saturated by the edge uv in $M(G)$ means $uv \in M(G)$.

Mirzakhah and Kiani in [18] gave some results about the signless Laplacian coefficients of a graph G and ordered unicyclic graphs with fixed girth based on the signless Laplacian coefficients. Motivated by this result, we characterize the graphs which have minimum signless Laplacian coefficients in $\mathcal{G}_{g_1}(n, m)$, $\mathcal{G}_{g_2}(n, m)$ and $\mathcal{G}_{g_1}(n)$, $\mathcal{G}_{g_2}(n)$. Let $G_g(s_1, t_1; s_2, t_2, \dots, s_g, t_g)$ denote the connected unicyclic graph of order n obtained from a cycle $C : u_1 u_2 \dots u_g u_1$ by adding s_i pendent paths of length 2 and t_i pendent edges at the vertex u_i ($i = 1, 2, \dots, g$).

This paper is organized as follows: In the next section, we introduce several transformations which simultaneously decrease all the signless Laplacian coefficients. In Section 3, we order all the n -vertex graphs in the sets $\mathcal{G}_3(s_1, t_1; s_2, t_2; s_3, t_3)$ and $\mathcal{G}_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)$ with given matching number m . In Section 4, by using the results of Section 2 and Section 3, we characterize the extremal graphs which have minimal signless Laplacian coefficients in $\mathcal{G}_{g_1}(n, m)$ (resp. $\mathcal{G}_{g_2}(n, m)$), as well as incidence energy. In Section 5, similar to the methods which are used in Section 4, we give the extremal graphs which have minimal signless Laplacian coefficients and incidence energy in the set $\mathcal{G}_{g_1}(n)$ (resp. $\mathcal{G}_{g_2}(n)$).

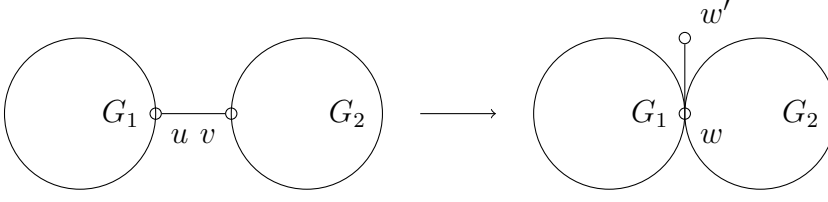


Figure 1: Transformation in Definition 2.1

2 Transformations

A pendent edge is an edge which is incident to a vertex of degree 1. A pendent path of length k attached to u is a path $uv_1v_2 \cdots v_k$, satisfying $d(v_1) = d(v_2) = \cdots = d(v_{k-1}) = 2, d(v_k) = 1$ and $d(u) \geq 3$. Let $N_G(v)$ denote the neighbors of v in the graph G .

For $v \in V(G)$, the subgraph $G \setminus \{v\}$ denotes the graph obtained by deleting the vertex v and all its incident edges. Denote the cycle of G as C , if the girth of G is g , write C as C_g . Assume $V(C_g) = \{u_1, u_2, \cdots, u_g\}$, after deleting all the edges of the cycle, g trees rooted at u_1, u_2, \cdots, u_g are obtained, denote them as $T_{u_1}^G, T_{u_2}^G, \cdots, T_{u_g}^G$.

For two vertices u, v in G , the distance $dist_G(u, v)$ of u, v equals the length of the shortest path in G . Let $dist_G(v, C)$ denote the distance between the vertex v and the cycle C in G , and $dist_G(v, C) = \min\{dist_G(u, v) : u \in C\}$. If $v \in T_{u_i}^G$, $dist_G(v, C) = dist_G(v, u_i)$. A branch vertex is a vertex with degree at least 3.

Definition 2.1 Let G be a simple connected graph with n vertices, and let uv be a nonpendent edge not contained in the cycle of G , let G_{uv} obtained from G by identifying vertices u and v and add a new pendent edge ww' to the new vertex w . (see fig.1).

Theorem 2.2 Let G be a n -vertex unicyclic graph, let G and G_{uv} be the two graphs presented in definition 2.1. Then

$$\varphi_i(G) \geq \varphi_i(G_{uv}), i = 0, 1, \cdots, n,$$

with equality if and only if either $i \in \{0, 1, n\}$ when G is non-bipartite, or $i \in \{0, 1, n-1, n\}$ otherwise.

Proof. From Theorem 1.2, according to the previous section, we have

$$\varphi_0(G) = \varphi_0(G_{uv}), \varphi_1(G) = \varphi_1(G_{uv}), \varphi_n(G) = \varphi_n(G_{uv}).$$

When G is bipartite graph, since this transformation does not change the length of the cycle, we have $\varphi_{n-1}(G) = \varphi_{n-1}(G_{uv})$.

When G is non-bipartite, for $2 \leq i \leq n-1$, denote \mathcal{H}'_i and \mathcal{H}_i the sets of all TU-subgraphs of G_{uv} and G with exactly i edges, respectively. For an arbitrary TU-subgraph $H' \in \mathcal{H}'_i$, let R' be the component of H' containing w . Let $N_{R'}(w) \cap N_G(u) = \{u_{i_1}, u_{i_2}, \dots, u_{i_r}\}$, where $0 \leq r \leq \min\{d_G(u) - 1, |V(R')| - 1\}$, $N_{R'}(w) \cap N_G(v) = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$, where $0 \leq s \leq \min\{d_G(v) - 1, |V(R')| - 1\}$. Define H with $V(H) = V(H') - \{w, w'\} + \{u, v\}$, if $ww' \notin E(H')$, $E(H) = E(H') - ww' - uu_{i_1} - \dots - uu_{i_r} - vv_{i_1} - \dots - vv_{i_s} + uu_{i_1} + \dots + uu_{i_r} + vv_{i_1} + \dots + vv_{i_s}$. If $ww' \in E(H')$, let $E(H) = E(H') - ww' - uu_{i_1} - \dots - uu_{i_r} - vv_{i_1} - \dots - vv_{i_s} + uu_{i_1} + \dots + uu_{i_r} + vv_{i_1} + \dots + vv_{i_s} + uv - ww'$. Let $f : \mathcal{H}'_i \rightarrow \mathcal{H}_i$, and $\mathcal{H}_i^* = f(\mathcal{H}'_i) = \{f(H') | H' \in \mathcal{H}'_i\}$.

Now we distinguish \mathcal{H}'_i into the following three cases. Denote G_1 the connected component containing u after deleting uv from G , and let G_2 be the connected component containing v after deleting uv from G .

Case 1: $ww' \in H'$, then H and H' have all the components of equal size, thus $W(H) = W(H')$.

Case 2: $ww' \notin H'$, w is in an odd unicyclic component U' of H' , By the symmetry of G_1 and G_2 , without loss of generality, assume G_1 contains an odd cycle as a subgraph. Assume U' contains a vertices in $G_2 \setminus \{w\}$ ($a \geq 0$), then $W(H') = 4 \cdot 1 \cdot N$, for some constant value N , $W(H) = 4 \cdot (a+1) \cdot N$. Thus $W(H) \geq W(H')$.

Case 3: $ww' \notin H'$, w is in a tree T' of H' . Assume T' contains b vertices in $G_1 \setminus \{w\}$ and c vertices in $G_2 \setminus \{w\}$, then $W(H') = (b+c+1) \cdot 1 \cdot N$, for some constant value N , $W(H) = (b+1) \cdot (c+1) \cdot N$. Thus $W(H) \geq W(H')$.

Therefore, by above discussions, $\varphi_i(G) > \varphi_i(G_{uv})$, $i = 2, \dots, n-1$ holds.

When G is bipartite, it is easy to prove $\varphi_i(G) > \varphi_i(G_{uv})$, $i = 2, \dots, n-2$ by using above discussions of Case 1 and Case 3. ■

Remark. When the subgraph induced by $V(G_2)$ is a star, it is easy to see that the result of Theorem 1.4 is a special case of Theorem 2.2.

When $E_{uv}^u \cap M(G) = \emptyset$, if $uv \in M(G)$, then $M(G_{uv}) = M(G) - \{uv\} + \{ww'\}$. If $vv_i \in M(G)$, for some $v_i \in G_2$, then $M(G_{uv}) = M(G) - \{vv_i\} + \{ww'\}$.

When $E_{uv}^v \cap M(G) = \emptyset$, the discussion is similar.

Thus if $E_{uv}^u \cap M(G) = \emptyset$ or $E_{uv}^v \cap M(G) = \emptyset$, we have $|M(G)| = |M(G_{uv})|$.

Definition 2.3 Let G be a n -vertex unicyclic graph, and let uv be a nonpendent edge of G not contained in the cycle, $d_G(u) \geq 3$, $d_G(v) \geq 2$ and uu' is a pendent edge. Let

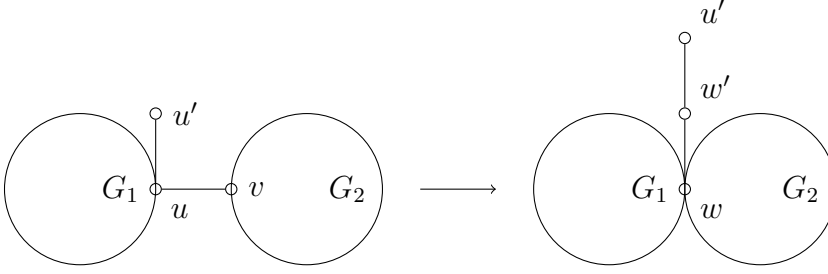


Figure 2: Transformation in Definition 2.3

G'_{uv} be the graph obtained by deleting vertex u' and edge uv , identifying u and v to a new vertex w , and adding a pendent path $ww'u'$ to the new vertex w . (see fig.2).

Theorem 2.4 Let G be a n -vertex unicyclic graph, let G and G'_{uv} be the two graphs presented in definition 2.3. Then

$$\varphi_i(G) \geq \varphi_i(G'_{uv}), i = 0, 1, \dots, n,$$

with equality if and only if either $i \in \{0, 1, n\}$ when G is non-bipartite, or $i \in \{0, 1, n-1, n\}$ otherwise.

Proof. For the case $\varphi_i(G) = \varphi_i(G'_{uv})$, the proof is similar to Theorem 2.2. Thus suppose G is non-bipartite and $2 \leq i \leq n-1$, denote \mathcal{H}'_i and \mathcal{H}_i the sets of all TU-subgraphs of G'_{uv} and G with exactly i edges, respectively. For an arbitrary TU-subgraph $H' \in \mathcal{H}'_i$, let R' be the component of H' containing w . Let $N_{R'}(w) \cap N_G(u) = \{u_{i_1}, u_{i_2}, \dots, u_{i_r}\}$, where $0 \leq r \leq \min\{d_G(u) - 2, |V(R')| - 1\}$, $N_{R'}(w) \cap N_G(v) = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$, where $0 \leq s \leq \min\{d_G(v) - 1, |V(R')| - 1\}$. Define H with $V(H) = V(H') - \{w, w'\} + \{u, v\}$, if $ww' \notin E(H')$, $E(H) = E(H') - ww_{i_1} - \dots - ww_{i_r} - ww_{i_1} - \dots - ww_{i_s} + uu_{i_1} + \dots + uu_{i_r} + vv_{i_1} + \dots + vv_{i_s}$. If $ww' \in E(H')$, let $E(H) = E(H') - ww_{i_1} - \dots - ww_{i_r} - ww_{i_1} - \dots - ww_{i_s} + uu_{i_1} + \dots + uu_{i_r} + vv_{i_1} + \dots + vv_{i_s} + uv - ww'$. Let $f: \mathcal{H}'_i \rightarrow \mathcal{H}_i$, and $\mathcal{H}_i^* = f(\mathcal{H}'_i) = \{f(H') | H' \in \mathcal{H}'_i\}$.

If we include w, w' in a component of H' , we have components of equal size in both TU-subgraphs (H and H'), then $W(H) = W(H')$. Now we can assume w, w' belong to two components of H' and distinguish \mathcal{H}'_i into the following three cases. Denote G_1 the subgraph containing u obtained by deleting uv and uu' from G , and let G_2 be the connected component containing v after deleting uv from G .

Case 1: w is in an odd unicyclic component U' of H' and the cycle is a subgraph of G_1 .

Subcase 1.1: $w'u' \notin H'$, assume U' contains b_1 vertices in the subgraph $G_2 \setminus \{w\}$, ($b_1 \geq 0$), then $W(H') = 4 \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = 4 \cdot (b_1 + 1) \cdot 1 \cdot N$, then $W(H) \geq W(H')$. Denote this set of H' as \mathcal{H}'_{11} .

Subcase 1.2: $w'u' \in H'$, assume U' contains b_2 vertices in the subgraph $G_2 \setminus \{w\}$, ($b_2 \geq 0$), then $W(H') = 4 \cdot 2 \cdot N$, for some constant value N . $W(H) = 4 \cdot (b_2 + 1) \cdot N$, then when $b_2 \geq 1$, $W(H) \geq W(H')$. When $b_2 = 0$, $W(H) < W(H')$, denote this set of H' as \mathcal{H}'_{12} .

Since the subgraph induced by $V(G_2)$ is a tree, and v is not a pendent vertex, $d(v) \geq 2$. Suppose one neighbor vertex of v except u is u_1 , $u_1 \in T' \subset H'$ and $|T'| = c \geq 1$. Then for every TU-subgraph $H'_2 \in \mathcal{H}'_{12}$, let $H'_1 = H'_2 - w'u' + wu_1$, $H_1 = H_2 - uu' + vu_1$, it is obvious that $H'_1 \in \mathcal{H}'_{11}$ and the mapping $f_1 : \mathcal{H}'_{12} \rightarrow \mathcal{H}'_{11}$ is an injection. Then $W(H'_2) = 4 \cdot 2 \cdot c \cdot N_1$, for some constant value N_1 . $W(H_2) = 4 \cdot (1 + c) \cdot N_1$, and $W(H'_1) = 4 \cdot 1 \cdot 1 \cdot N_1$, $W(H_1) = 4 \cdot 1 \cdot (1 + c) \cdot N_1$. Therefore,

$$\begin{aligned} & \sum_{H'_1 \in \mathcal{H}'_{11}} [W(H_1) - W(H'_1)] + \sum_{H'_2 \in \mathcal{H}'_{12}} [W(H_2) - W(H'_2)] \\ & \geq \sum_{H'_2 \in \mathcal{H}'_{12}} (4 + 4c - 8c) \cdot N_1 + \sum_{H'_1 \in f_1(\mathcal{H}'_{12})} (4c) \cdot N_1 \\ & = 4 \sum_{H'_2 \in \mathcal{H}'_{12}} N_1 > 0 \end{aligned}$$

Case 2: w is in an odd unicyclic component U' of H' and the cycle is a subgraph of G_2 .

Subcase 2.1: $w'u' \notin H'$, assume U' contains a_1 vertices in the subgraph $G_1 \setminus \{w\}$, ($a_1 \geq 0$), then $W(H') = 4 \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = 4 \cdot (a_1 + 1) \cdot 1 \cdot N$, then $W(H) \geq W(H')$. Denote this set of H' as \mathcal{H}'_{21} .

Subcase 2.2: $w'u' \in H'$, assume U' contains a_2 vertices in the subgraph $G_1 \setminus \{w\}$, ($a_2 \geq 0$), then $W(H') = 4 \cdot 2 \cdot N$, for some constant value N . $W(H) = 4 \cdot (a_2 + 2) \cdot N$, then $W(H) \geq W(H')$.

Case 3: w is in a tree T' of H' .

Subcase 3.1: $w'u' \notin H'$, $w \in T' \subset H'$ and T' contains m_1 vertices in the subgraph $G_1 \setminus \{w\}$, ($m_1 \geq 0$), and n_1 vertices in the subgraph $G_2 \setminus \{w\}$, ($n_1 \geq 0$). Then $W(H') = (m_1 + n_1 + 1) \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = (m_1 + 1) \cdot (n_1 + 1) \cdot 1 \cdot N$, then $W(H) \geq W(H')$. Denote this set of H' as \mathcal{H}'_{31} .

Subcase 3.2: $w'u' \in H'$, $w \in T' \subset H'$ and T' contains m_2 vertices in the subgraph $G_1 \setminus \{w\}$, ($m_2 \geq 0$), and n_2 vertices in the subgraph $G_2 \setminus \{w\}$, ($n_2 \geq 0$). Then $W(H') =$

$(m_2 + n_2 + 1) \cdot 2 \cdot N$, for some constant value N . $W(H) = (m_2 + 2) \cdot (n_2 + 1) \cdot N$, then $W(H) - W(H') = m_2 \cdot (n_2 - 1) \cdot N$. If $m_2 = 0$ or $m_2 > 0, n_2 \geq 1$, $W(H) \geq W(H')$ holds. If $m_2 > 0, n_2 = 0$, the inequality $W(H) < W(H')$ is obtained. Denote this set of H' as \mathcal{H}'_{32} .

Now we discuss the subcase in which $H' \in \mathcal{H}'_{32}$. Since v is not pendent vertex, $d(v) \geq 2$, assume there is at least one neighbor vertex of w in $V(G_2)$, denoted by v_1 , which satisfies $v_1 \in T'_1$, $|T'_1| = p$, ($p \geq 1$). Denote the set of H' of this kind as $\mathcal{H}'^{(1)}_{32}$. For every TU-subgraph $H'_1 \in \mathcal{H}'^{(1)}_{32}$, let $H'_2 = H'_1 - w'u' + wv_1$, $H_2 = H_1 - uu' + vv_1$. It is obvious that $H'_2 \in \mathcal{H}'_{31}$ and the mapping $f_2 : \mathcal{H}'^{(1)}_{32} \rightarrow \mathcal{H}'_{31}, \mathcal{H}'^{(1)}_{32} \rightarrow \mathcal{H}_{31}$ is an injection. Then $W(H'_1) = (m_1 + 1) \cdot 2 \cdot p \cdot N$, for some constant value N . $W(H_1) = (m_1 + 2) \cdot 1 \cdot p \cdot N$, and $W(H'_2) = (m_1 + 1 + p) \cdot 1 \cdot 1 \cdot N$, $W(H_2) = (m_1 + 1) \cdot 1 \cdot (p + 1) \cdot N$. Then

$$\begin{aligned} & \sum_{H'_1 \in \mathcal{H}'^{(1)}_{32}} [W(H_1) - W(H'_1)] + \sum_{H'_2 \in \mathcal{H}'_{31}} [W(H_2) - W(H'_2)] \\ & \geq \sum_{H'_1 \in \mathcal{H}'^{(1)}_{32}} (-p \cdot m_1) \cdot N + \sum_{H'_2 \in f_2(\mathcal{H}'^{(1)}_{32})} p \cdot m_1 \cdot N = 0. \end{aligned}$$

If all the neighbors of w in $V(G_2)$ are in an odd unicyclic component U' , then denote the set of H' of this kind as $\mathcal{H}'^{(2)}_{32}$. For every TU-subgraph $H'_1 \in \mathcal{H}'^{(2)}_{32}$, let $H'_2 = H'_1 - w'u' + wv_1$, $H_2 = H_1 - uu' + vv_1$, it is obvious that $H'_2 \in \mathcal{H}'_{21}$ and the mapping $f_3 : \mathcal{H}'^{(2)}_{32} \rightarrow \mathcal{H}'_{21}, \mathcal{H}'^{(2)}_{32} \rightarrow \mathcal{H}_{21}$ is an injection. Then $W(H'_1) = 4 \cdot 2 \cdot (m_1 + 1) \cdot N$, for some constant value N . $W(H_1) = 4 \cdot 1 \cdot (m_1 + 2) \cdot N$, and $W(H'_2) = 4 \cdot 1 \cdot 1 \cdot N$, $W(H_2) = 4 \cdot 1 \cdot (m_1 + 1) \cdot N$. Then

$$\begin{aligned} & \sum_{H'_1 \in \mathcal{H}'^{(2)}_{32}} [W(H_1) - W(H'_1)] + \sum_{H'_2 \in \mathcal{H}'_{21}} [W(H_2) - W(H'_2)] \\ & \geq \sum_{H'_1 \in \mathcal{H}'^{(2)}_{32}} (-4 \cdot m_1 \cdot N) + \sum_{H'_2 \in f_3(\mathcal{H}'^{(2)}_{32})} (4 \cdot m_1 \cdot N) = 0 \end{aligned}$$

It is easy to see that the correspondence from H' to H defined above is an injection. By summing over possible TU-subsets of i edges of G'_{uv} , from Theorem 1.1, the results is obtained.

When G is bipartite, it is easy to prove $\varphi_i(G) > \varphi_i(G_{uv}), i = 2, \dots, n - 2$ by using above discussions of Case 3. ■

Remark. Assume $uu' \in M(G)$.

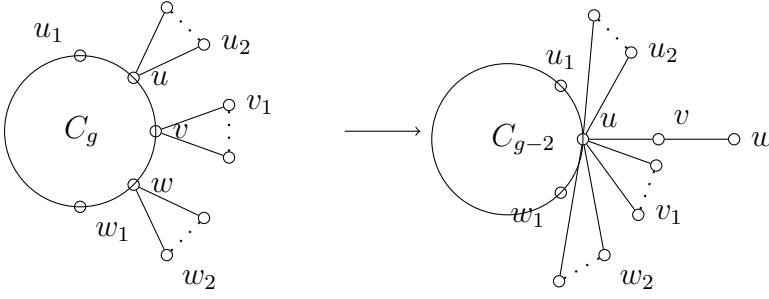


Figure 3: The Transformation of Definition 2.5

If $E_{uv}^v \cap M(G) = \{vv_j\}$ for some $v_j \in G_2$, then $M(G'_{uv}) = M(G) - \{uu', vv_j\} + \{w'u', wv_j\}$.

If $E_{uv}^v \cap M(G) = \emptyset$ and all neighbors of u in G_1 are saturated in $M(G)$, then $M(G'_{uv}) = M(G) - \{uu'\} + \{w'u'\}$.

If $E_{uv}^v \cap M(G) = \emptyset$ and there is one neighbor u_i of u in G_1 which is not saturated in $M(G)$, then $M(G'_{uv}) = M(G) - \{uu'\} + \{w'u', wu_i\}$.

Thus, after the transformation of Definition 2.3, $|M(G'_{uv})| = |M(G)|$ or $|M(G)| + 1$.

Definition 2.5 Let G be a n -vertex unicyclic graph with girth g , $n \geq 6$, there are only pendent paths of length at most 2 attached to the cycle C_g . u, v, w are on the cycle of length at least 5 and $u \sim v, v \sim w$. (see fig.3). Assume $N_G(u) = \{v, u_1, u_2, \dots\}$, $N_G(v) = \{u, w, v_1, v_2, \dots\}$, $N_G(w) = \{v, w_1, w_2, \dots\}$, then the graph

$$G' = G - \{ww_1, ww_2, \dots, vv_1, vv_2, \dots\} + \{uw_1, uw_2, \dots, uv_1, uv_2, \dots\}$$

Theorem 2.6 Let G and G' be the two graphs presented in Definition 2.5, and the length of the cycle of G is g . Then

$$\varphi_i(G) \geq \varphi_i(G'), i = 0, 1, \dots, n,$$

with equality if and only if $i \in \{0, 1, n\}$.

Proof. For $i \in \{0, 1, n\}$, the proof is similar to Theorem 2.2. Thus suppose $2 \leq i \leq n - 1$, denote \mathcal{H}'_i and \mathcal{H}_i the sets of all TU-subgraphs of G' and G with exactly i edges, respectively.

First assume g is odd. For an arbitrary TU-subgraph $H' \in \mathcal{H}'_i$, let R' be the component of H' containing u . Let $N_{R'}(u) \cap N_G(w) = \{w_{i_1}, w_{i_2}, \dots, w_{i_r}\}$, where

$0 \leq r \leq \min\{d_G(w) - 1, |V(R')| - 1\}$, $N_{R'}(u) \cap N_G(v) = \{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$, where $0 \leq s \leq \min\{d_G(v) - 2, |V(R')| - 1\}$. Define H with $V(H) = V(H')$, $E(H) = E(H') - uw_{i_1} - \dots - uw_{i_r} - uv_{i_1} - \dots - uv_{i_s} + ww_{i_1} + \dots + ww_{i_r} + vv_{i_1} + \dots + vv_{i_s}$. Denote U and U' be the connected component of H and H' containing an odd cycle, respectively. Let $f : \mathcal{H}'_i \rightarrow \mathcal{H}_i$, and $\mathcal{H}_i^* = f(\mathcal{H}'_i) = \{f(H') | H' \in \mathcal{H}'_i\}$.

For convenience, write \mathcal{H}'_i as \mathcal{H}' , and \mathcal{H}_i as \mathcal{H} . If we include u, v, w in a component of H' , then we have components of equal sizes in both TU-subgraphs H' and H , and thus $W(H) = W(H')$ in these cases. Denote $\mathcal{H}^{(0)} = \{H' | uv \in H', vw \in H'\}$. Now we can assume that u, v, w belong to 2 or 3 components.

We distinguish \mathcal{H}' into the following two cases.

Case 1: u is not in an odd unicyclic component of H' , then all components of H' are trees. Assume $u \in T'_1$, and there are a_1 vertices in $V(T'_u) \cap V(T'_1)$ and the vertices in the counter-clockwise of u (excluding u), and a_3 vertices in the set $V(T'_v) \cap V(T'_1)$ (excluding v), a_2 vertices among the set $V(T'_w) \cap V(T'_1)$ and the vertices in the clockwise of u (excluding w) ($a_1, a_2, a_3 \geq 0$). Actually in this case, $a_3 = s$. Denote N the product of all the orders of components of H' except the components containing u, v, w .

Subcase 1.1: $uv \in H', vw \notin H'$, then $W(H') = (a_1 + a_2 + a_3 + 2) \cdot 1 \cdot N$, for some constant value N . $W(H) = (a_1 + a_3 + 2) \cdot (a_2 + 1) \cdot N$, so $W(H) - W(H') = [a_2 \cdot (a_1 + a_3 + 1)] \cdot N \geq 0$. Denote $\mathcal{H}'^{(11)} = \{H' | u \in T'_1, uv \in H', vw \notin H', a_1 = 0\}$, $\mathcal{H}'^{(12)} = \{H' | u \in T'_1, uv \in H', vw \notin H', a_1 \geq 1\}$. Then $\sum_{H' \in \mathcal{H}'^{(11)}} [W(H) - W(H')] \geq 0$, and $\sum_{H' \in \mathcal{H}'^{(12)}} [W(H) - W(H')] \geq 0$.

Subcase 1.2: $uv, vw \notin H'$, $W(H') = (a_1 + a_2 + a_3 + 1) \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = (a_1 + 1) \cdot (a_2 + 1) \cdot (a_3 + 1) \cdot N$, so $W(H) - W(H') \geq 0$. Denote $\mathcal{H}'^{(21)} = \{H' | u \in T'_1, uv, vw \notin H', a_1 = 0 \text{ or } a_2 \geq 1\}$, $\mathcal{H}'^{(22)} = \{H' | u \in T'_1, uv, vw \notin H', a_1 \geq 1, a_2 = 0\}$. Then $\sum_{H' \in \mathcal{H}'^{(21)}} [W(H) - W(H')] \geq 0$, and $\sum_{H' \in \mathcal{H}'^{(22)}} [W(H) - W(H')] \geq 0$.

Subcase 1.3: $uv \notin H', vw \in H'$, then $W(H') = (a_1 + a_2 + a_3 + 1) \cdot 2 \cdot N$, for some constant value N . $W(H) = (a_1 + 1) \cdot (a_2 + a_3 + 2) \cdot N$, then $W(H) - W(H') = (a_1 - 1) \cdot (a_2 + a_3) \cdot N$.

Denote $\mathcal{H}'^{(30)} = \{H' | u \in T'_1, uv \notin H', vw \in H', a_1 \geq 1\}$, $\mathcal{H}'^{(31)} = \{H' | u \in T'_1, uv \notin H', vw \in H', a_1 = 0, a_2 \geq 1\}$, $\mathcal{H}'^{(32)} = \{H' | u \in T'_1, uv \notin H', vw \in H', a_1 = a_2 = 0\}$. then

$$\sum_{H' \in \mathcal{H}'^{(30)}} [W(H) - W(H')] \geq 0,$$

and for every TU-subgraph $H'^{(31)} \in \mathcal{H}'^{(31)}$, let $H'^{(11)} = H'^{(31)} - vw + uv$, $H^{(11)} = H^{(31)} - vw + uv$, it is obvious that $H'^{(11)} \in \mathcal{H}'^{(11)}$, and the mapping $f_1 : \mathcal{H}'^{(31)} \rightarrow \mathcal{H}'^{(11)}, \mathcal{H}^{(31)} \rightarrow \mathcal{H}^{(11)}$ is an injection. Then $W(H'^{(31)}) = (a_2 + a_3 + 1) \cdot 2 \cdot N$, for some constant value N . $W(H^{(31)}) = (a_2 + a_3 + 2) \cdot 1 \cdot N$, and $W(H'^{(11)}) = (a_2 + a_3 + 2) \cdot 1 \cdot N$, $W(H^{(11)}) = (a_2 + 1) \cdot (a_3 + 2) \cdot N$. Then

$$\begin{aligned}
& \left(\sum_{H' \in \mathcal{H}'^{(31)}} + \sum_{H' \in \mathcal{H}'^{(11)}} \right) [W(H) - W(H')] \\
&= \sum_{H'^{(31)} \in \mathcal{H}'^{(31)}} [W(H^{(31)}) - W(H'^{(31)})] + \sum_{H'^{(11)} \in \mathcal{H}'^{(11)}} [W(H^{(11)}) - W(H'^{(11)})] \\
&\geq \sum_{H'^{(31)} \in \mathcal{H}'^{(31)}} [-(a_2 + a_3) \cdot N] + \sum_{H'^{(11)} \in f_1(\mathcal{H}'^{(31)})} [(a_2 \cdot a_3 + a_2) \cdot N] \\
&= \sum_{H'^{(31)} \in \mathcal{H}'^{(31)}} [(a_2 - 1) \cdot a_3 \cdot N] \geq 0.
\end{aligned}$$

For every TU-subgraph $H'^{(32)} \in \mathcal{H}'^{(32)}$, $uu_1 \notin H'^{(32)}$ and assume u_1 is in a tree of order c in $H'^{(32)}$. Let $H'^{(22)} = H'^{(32)} - vw + uu_1$, $H^{(22)} = H^{(32)} - vw + uu_1$, it is obvious that $H'^{(22)} \in \mathcal{H}'^{(22)}$ and the mapping $f_2 : \mathcal{H}'^{(32)} \rightarrow \mathcal{H}'^{(22)}, \mathcal{H}^{(32)} \rightarrow \mathcal{H}^{(22)}$ is an injection. Then $W(H'^{(32)}) = (a_3 + 1) \cdot 2 \cdot c \cdot N$, for some constant value N . $W(H^{(32)}) = (a_3 + 2) \cdot 1 \cdot c \cdot N$, and $W(H'^{(22)}) = (a_3 + 1 + c) \cdot 1 \cdot 1 \cdot N$, $W(H^{(22)}) = (a_3 + 1) \cdot (1 + c) \cdot 1 \cdot N$. Then

$$\begin{aligned}
& \left(\sum_{H' \in \mathcal{H}'^{(32)}} + \sum_{H' \in \mathcal{H}'^{(22)}} \right) [W(H) - W(H')] \\
&= \sum_{H'^{(32)} \in \mathcal{H}'^{(32)}} [W(H^{(32)}) - W(H'^{(32)})] + \sum_{H'^{(22)} \in \mathcal{H}'^{(22)}} [W(H^{(22)}) - W(H'^{(22)})] \\
&\geq \sum_{H'^{(32)} \in \mathcal{H}'^{(32)}} (-a_3 \cdot c \cdot N) + \sum_{H'^{(22)} \in f_2(\mathcal{H}'^{(32)})} (a_3 \cdot c \cdot N) = 0.
\end{aligned}$$

Case 2: u is in an odd unicyclic component U' of H' . Assume the number of vertices of U' which are incident to the vertices of the cycle of G' is $d(d \geq 0)$.

Subcase 2.1: $uv \notin H', vw \notin H'$, then $W(H') = 4 \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) \geq (g - 1) \cdot 1 \cdot N$, so $W(H) - W(H') \geq (g - 5) \cdot N \geq 0$. Denote $\mathcal{H}'^{(4)} = \{H' | u \in U', uv \notin H', vw \notin H'\}$, then $\sum_{H' \in \mathcal{H}'^{(4)}} [W(H) - W(H')] \geq 0$.

Subcase 2.2: $uv \notin H', vw \in H'$, then $W(H') = 4 \cdot 2 \cdot N$, for some constant value N . $W(H) \geq (g + d) \cdot N \geq g \cdot N$, then $W(H) - W(H') \geq (g + d - 8) \cdot N \geq (g - 8) \cdot N$.

Denote $\mathcal{H}'^{(50)} = \{H' | u \in U', uv \notin H', vw \in H', d = 0, g = 5\}$, and $\mathcal{H}'^{(51)} = \{H' | u \in U', uv \notin H', vw \in H', d \geq 1 \text{ or } g \geq 6\}$.

Subcase 2.3: $uv \in H', vw \notin H'$, then $W(H') = 4 \cdot 1 \cdot N$, for some constant value N . $W(H) \geq (g + d) \cdot N \geq g \cdot N$, so $W(H) - W(H') \geq (g + d - 4) \cdot N \geq (g - 4) \cdot N$. Denote $\mathcal{H}'^{(60)} = \{H' | u \in U', uv \in H', vw \notin H', d = 0, g = 5\}$, and $\mathcal{H}'^{(61)} = \{H' | u \in U', uv \in H', vw \notin H', d \geq 1 \text{ or } g \geq 6\}$.

For every $H'^{(61)} \in \mathcal{H}'^{(61)}$, set $H'^{(51)} = H'^{(61)} - uv + vw$. It is obvious that $H'^{(51)} \in \mathcal{H}'^{(51)}$ and $f_3 : \mathcal{H}'^{(61)} \rightarrow \mathcal{H}'^{(51)}$ is an injection. For every $H'^{(51)} \in \mathcal{H}'^{(51)}$, set $H'^{(61)} = H'^{(51)} - vw + uv$. It is obvious that $H'^{(61)} \in \mathcal{H}'^{(61)}$ and the mapping $g_3 : \mathcal{H}'^{(51)} \rightarrow \mathcal{H}'^{(61)}$ is an injection. Thus $g_3 = f_3^{-1}$ and $f_3 : \mathcal{H}'^{(61)} \rightarrow \mathcal{H}'^{(51)}$ is a bijection.

$$\text{Then } \sum_{H' \in \mathcal{H}'^{(61)}} (W(H) - W(H')) + \sum_{H' \in \mathcal{H}'^{(51)}} (W(H) - W(H')) \geq$$

$$\sum_{H' \in \mathcal{H}'^{(61)}} ((g + d - 4) \cdot N + (g + d - 8) \cdot N) = \sum_{H' \in \mathcal{H}'^{(61)}} (2 \cdot (g + d - 6) \cdot N) \geq 0.$$

When $H' \in \mathcal{H}'^{(60)}$ or $H' \in \mathcal{H}'^{(50)}$, we have $g = 5, d = 0$ and TU-subgraph H' and H for Subcase 2.2 and 2.3 have 4 edges.

If $|E(H')| = 4$, since there is at least one pendent path attached to u_1 or w_1 or u in G' , without loss of generality, let $u_0 \in N_{G'}(u) \setminus \{v, u_1, w_1\}$.

For every $H'^{(60)} \in \mathcal{H}'^{(60)}$ and $H'^{(50)} \in \mathcal{H}'^{(50)}$, set $H'^{(12)} = H'^{(60)} - u_1 w_1 + u u_0$ and $H'^{(30)} = H'^{(50)} - u_1 w_1 + u u_0$. It is obvious that $H'^{(12)} \in \mathcal{H}'^{(12)}$, $H'^{(30)} \in \mathcal{H}'^{(30)}$, and $h_1 : \mathcal{H}'^{(60)} \rightarrow \mathcal{H}'^{(12)}$ is an injection, $h_2 : \mathcal{H}'^{(50)} \rightarrow \mathcal{H}'^{(30)}$ is an injection. Furthermore, by the previous discussion, it is easy to prove that $f_4 : \mathcal{H}'^{(60)} \rightarrow \mathcal{H}'^{(50)}$ is a bijection, and $(\sum_{H' \in \mathcal{H}'^{(60)}} + \sum_{H' \in \mathcal{H}'^{(50)}})(W(H) - W(H')) \geq$

$$\sum_{H' \in \mathcal{H}'^{(60)}} ((g + d - 4) \cdot N + (g + d - 8) \cdot N) = \sum_{H' \in \mathcal{H}'^{(60)}} ((-2) \cdot N).$$

$$\text{Then } (\sum_{H' \in \mathcal{H}'^{(50)}} + \sum_{H' \in \mathcal{H}'^{(60)}} + \sum_{H' \in \mathcal{H}'^{(30)}} + \sum_{H' \in \mathcal{H}'^{(12)}})(W(H) - W(H'))$$

$$\geq \sum_{H' \in \mathcal{H}'^{(60)}} [(-3 + 1 + 3 + 0) \cdot N] > 0$$

If $5 \leq |E(H')| \leq n - 1$, then there is at least one pendent edge which belongs to $E(H')$, without loss of generality, assume $u_0 u'_0 \in E(H')$, where $u_0 \in N_{G'}(u) \setminus \{v, u_1, w_1\}$. Then by similar method of the above case, we have

$$\begin{aligned} & (\sum_{H' \in \mathcal{H}'^{(60)}} + \sum_{H' \in \mathcal{H}'^{(50)}})(W(H) - W(H')) \\ & \geq \sum_{H' \in \mathcal{H}'^{(60)}} ((g + d - 4) \cdot N + (g + d - 8) \cdot N) \\ & = \sum_{H' \in \mathcal{H}'^{(60)}} (-2) \cdot 2 \cdot N/2 \geq 0, \end{aligned}$$

$$\begin{aligned} & \text{and } (\sum_{H' \in \mathcal{H}'(50)} + \sum_{H' \in \mathcal{H}'(60)} + \sum_{H' \in \mathcal{H}'(30)} + \sum_{H' \in \mathcal{H}'(12)})(W(H) - W(H')) \\ & \geq \sum_{H' \in \mathcal{H}'(60)} [(-4 + 4 + 0) \cdot N/2] \geq 0. \end{aligned}$$

Thus by summing over all possible subsets of \mathcal{H}'_i , ($\mathcal{H}'_i = \mathcal{H}'^{(0)} \cup \mathcal{H}'^{(11)} \cup \mathcal{H}'^{(12)} \cup \mathcal{H}'^{(21)} \cup \mathcal{H}'^{(22)} \cup \mathcal{H}'^{(30)} \cup \mathcal{H}'^{(31)} \cup \mathcal{H}'^{(32)} \cup \mathcal{H}'^{(4)} \cup \mathcal{H}'^{(50)} \cup \mathcal{H}'^{(51)} \cup \mathcal{H}'^{(60)} \cup \mathcal{H}'^{(61)}$), from Theorem 1.2 and f is an injection on the whole. Then

$$\varphi_i(G') = \sum_{H' \in \mathcal{H}'_i} W(H') < \sum_{H \in \mathcal{H}_i^*} W(H) \leq \sum_{H \in \mathcal{H}_i} W(H) = \varphi_i(G)$$

holds for $i = 2, 3, \dots, n-1$.

When g is even, the result $\varphi_i(G) > \varphi_i(G')$ holds for $i = 2, 3, \dots, n-1$, which can be proved as Case 1. ■

Remark. Assume there are t_i (resp. t_{i+1}, t_{i+2}) pendent edges and s_i (resp. s_{i+1}, s_{i+2}) pendent paths of length 2 attached to u (resp. v, w).

If $t_i, t_{i+1}, t_{i+2} \neq 0$, assume $uu', vv', ww' \in M(G)$, then

$$M(G') = M(G) - \{uu', vv', ww'\} + \{uu', vw\},$$

we have $M(G') = M(G) - 1$.

If $t_i = t_{i+1} = 0$, then $uv \in M(G)$, we have $M(G') = M(G) - \{uv\} + \{vw\}$, and $M(G') = M(G)$.

If $t_i, t_{i+1} \neq 0$, $t_{i+2} = 0$, assume $uu', vv' \in M(G)$, then $M(G') = M(G) - \{uu', vv'\} + \{uu', vw\}$, we have $M(G') = M(G)$. The case $t_{i+1}, t_{i+2} \neq 0$, $t_i = 0$ is similar.

If $t_{i+1} \neq 0$, $t_i = t_{i+2} = 0$, assume $vv' \in M(G)$, then $M(G') = M(G) - \{vv'\} + \{uv', vw\}$, we have $M(G') = M(G) + 1$.

Definition 2.7 Let G be a n -vertex unicyclic graph with girth g , $n \geq 8$, there are only pendent paths of lengths 1 or 2 attached to the cycle C_g . u, v, w are on the cycle of length at least 5 and there is at least one pendent edge attached to u, v, w , respectively. (see fig.4). Assume $u \sim v, v \sim w$ and $N_G(u) = \{v, u', u_2, u_3, \dots\}$, $N_G(v) = \{u, w, v', v_2, \dots\}$, $N_G(w) = \{v, w', w_2, w_3, \dots\}$, uu', vv', ww' are pendent edges of G . Then the graph

$$G' = G - \{vw, ww_2, ww_3, \dots, vv_2, vv_3, \dots\} + \{uw, uw_2, uw_3, \dots, uv_2, uv_3, \dots\}$$

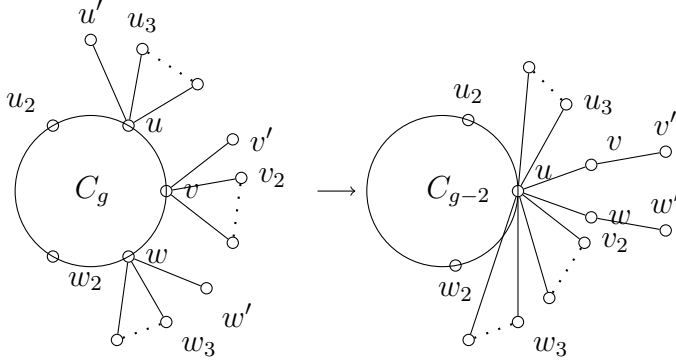


Figure 4: Transformation of Definition 2.7

Theorem 2.8 *Let G and G' be the two graphs presented in Definition 2.7, and the length of the cycle of G is g , $n \geq 8$. Then*

$$\varphi_i(G) \geq \varphi_i(G'), i = 0, 1, \dots, n,$$

with equality if and only if $i \in \{0, 1, n\}$.

Proof. For $n = 8$, there is a unique graph G which satisfies Theorem 2.8, by mathematical computing directly, $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n - 2$ and $\varphi_i(G) = \varphi_i(G'), i = 0, 1, n - 1, n$.

Next assume $n \geq 9$. For $i \in \{0, 1, n\}$, the proof is similar to Theorem 2.2. Thus suppose $2 \leq i \leq n - 1$, denote \mathcal{H}'_i and \mathcal{H}_i the sets of all TU-subgraphs of G' and G with exactly i edges, respectively.

First assume g is odd, for an arbitrary TU-subgraph $H' \in \mathcal{H}'_i$, let R' be the component of H' containing u . Denote $E'_1 = \{uv_i : uv_i \in E(R'), 2 \leq i \leq s\}, E'_2 = \{uw_i : uw_i \in E(R'), 2 \leq i \leq t\} \cup \{uw : uw \in E(R')\}. E_1 = \{vv_i : uv_i \in E'_1, 2 \leq i \leq s\}, E_2 = \{ww_i : uw_i \in E'_2, 2 \leq i \leq t\} \cup \{vw : uw \in E'_2\}$. Define H with $V(H) = V(H'), E(H) = E(H') - E'_1 - E'_2 + E_1 + E_2$. Let $f : \mathcal{H}'_i \rightarrow \mathcal{H}_i$, and $\mathcal{H}^*_i = f(\mathcal{H}'_i) = \{f(H') | H' \in \mathcal{H}'_i\}$.

If we include u, v, w in a component of H' , then we have components of equal sizes in both TU-subgraphs H' and H , and thus $W(H) = W(H')$ in this case. Denote $\mathcal{H}'_{(0)} = \{H' | uv \in H', vw \in H'\}$. Now we can assume that u, v, w belong to 2 or 3 components.

Next we distinguish \mathcal{H}' into the following two cases.

Case 1: u is not in an odd unicyclic component of H' , then all components of H' are trees. Assume $u \in T'_1$, and T'_1 contains b_1 vertices among the set $V(T_u^G) \setminus \{u'\}$

and the vertices in the counter-clockwise of u (excluding u), and b_3 vertices in the set $V(T_v^G) \setminus \{v'\}$ (excluding v), b_2 vertices among the set $V(T_w^G) \setminus \{w'\}$ and the vertices in the clockwise of u (excluding w) ($b_1, b_2, b_3 \geq 0$).

Subcase 1.1: $uv \in H', vv', uw, ww' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 2) \cdot 1 \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = (b_1 + b_3 + 2) \cdot (b_2 + 1) \cdot 1 \cdot 1 \cdot N$, so $W(H) - W(H') = b_2 \cdot (b_1 + b_3 + 1) \cdot N \geq 0$. Denote $\mathcal{H}'_{1,1} = \{H' | u \in T'_1, uv \in H', vv', uw, ww' \notin H'\}$.

Subcase 1.2: $uv, vv' \in H', uw, ww' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 3) \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = (b_1 + b_3 + 3) \cdot (b_2 + 1) \cdot 1 \cdot N$, so $W(H) - W(H') = b_2 \cdot (b_1 + b_3 + 2) \cdot N \geq 0$. Denote $\mathcal{H}'_{1,2} = \{H' | u \in T'_1, uv, vv' \in H', uw, ww' \notin H'\}$.

Subcase 1.3: $uv, ww' \in H', uw, vv' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 2) \cdot 1 \cdot 2 \cdot N$, for some constant value N . $W(H) = (b_1 + b_3 + 2) \cdot (b_2 + 2) \cdot 1 \cdot N$, so $W(H) - W(H') = b_2 \cdot (b_1 + b_3) \cdot N \geq 0$. Denote $\mathcal{H}'_{1,3} = \{H' | u \in T'_1, uv, ww' \in H', uw, vv' \notin H'\}$.

Subcase 1.4: $uv, ww', vv' \in H', uw \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 3) \cdot 2 \cdot N$, for some constant value N . $W(H) = (b_1 + b_3 + 3) \cdot (b_2 + 2) \cdot N$, so $W(H) - W(H') = b_2 \cdot (b_1 + b_3 + 1) \cdot N \geq 0$. Denote $\mathcal{H}'_{1,4} = \{H' | u \in T'_1, uv, ww', vv' \in H', uw \notin H'\}$.

Subcase 1.5: $uw \in H', uv, ww', vv' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 2) \cdot 1 \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = (b_1 + 1) \cdot (b_2 + b_3 + 2) \cdot 1 \cdot 1 \cdot N$, so $W(H) - W(H') = b_1 \cdot (b_2 + b_3 + 1) \cdot N \geq 0$. Denote $\mathcal{H}'_{1,5} = \{u \in T'_1, H' | uw \in H', uv, ww', vv' \notin H'\}$.

Subcase 1.6: $uw, vv' \in H', uv, ww' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 2) \cdot 2 \cdot 1 \cdot N$, for some constant value N . $W(H) = (b_1 + 1) \cdot (b_2 + b_3 + 3) \cdot 1 \cdot N$, so $W(H) - W(H') = (b_1 - 1) \cdot (b_2 + b_3 + 1) \cdot N$. Denote $\mathcal{H}'_{1,6}^{(1)} = \{H' | u \in T'_1, uw, vv' \in H', uv, ww' \notin H', b_1 \geq 1\}$. If $H' \in \mathcal{H}'_{1,6}^{(1)}$, $W(H) - W(H') \geq 0$. Denote $\mathcal{H}'_{1,6}^{(2)} = \{H' | u \in T'_1, uw, vv' \in H', uv, ww' \notin H', b_1 = 0\}$. For every $H'_1 \in \mathcal{H}'_{1,6}^{(2)}$, assume u_2 is in a component of H'_1 of order p . Set $H'_2 = H'_1 - vv' + uu_2$, it is obvious that $H'_2 \in \mathcal{H}'_{1,5}$ in which $b_1 = p \geq 1$ and $f_1 : \mathcal{H}'_{1,6}^{(2)} \rightarrow \mathcal{H}'_{1,5}$ is an injection. Then

$$\left(\sum_{H' \in \mathcal{H}'_{1,6}^{(2)}} + \sum_{H' \in \mathcal{H}'_{1,5}} \right) (W(H) - W(H')) \geq \left(\sum_{H' \in \mathcal{H}'_{1,6}^{(2)}} + \sum_{H' \in f_1(\mathcal{H}'_{1,6}^{(2)})} \right) (W(H) - W(H')) = 0.$$

Subcase 1.7: $uw, ww' \in H', uv, vv' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 3) \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = (b_1 + 1) \cdot (b_2 + b_3 + 3) \cdot 1 \cdot 1 \cdot N$, so $W(H) - W(H') = b_1 \cdot (b_2 + b_3 + 2) \cdot N \geq 0$. Denote $\mathcal{H}'_{1,7} = \{H' | u \in T'_1, uw, ww' \in H', uv, vv' \notin H'\}$.

Subcase 1.8: $uw, vv', ww' \in H', uv \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 3) \cdot 2 \cdot N$, for some constant value N . $W(H) = (b_1 + 1) \cdot (b_2 + b_3 + 4) \cdot N$, so $W(H) - W(H') = (b_1 - 1) \cdot (b_2 + b_3 + 2) \cdot N$. Denote $\mathcal{H}'_{1,8}^{(1)} = \{H' | u \in T'_1, uw, vv', ww' \in H', uv \notin H', b_1 \geq 1\}$. If $H' \in \mathcal{H}'_{1,8}^{(1)}$, $W(H) - W(H') \geq 0$. Denote $\mathcal{H}'_{1,8}^{(2)} = \{H' | u \in T'_1, uw, vv', ww' \in H', uv \notin H', b_1 = 0\}$.

$H', b_1 = 0\}$. For every $H'_1 \in \mathcal{H}'_{1,8}{}^{(2)}$, assume u_2 is in a component of H'_1 of order p . Set $H'_2 = H'_1 - vv' + uu_2$, it is obvious that $H'_2 \in \mathcal{H}'_{1,7}$ in which $b_1 = p \geq 1$ and $f_2 : \mathcal{H}'_{1,8}{}^{(2)} \rightarrow \mathcal{H}'_{1,7}$ is an injection. Then

$$\left(\sum_{H' \in \mathcal{H}'_{1,8}{}^{(2)}} + \sum_{H' \in \mathcal{H}'_{1,7}} \right) (W(H) - W(H')) \geq \left(\sum_{H' \in \mathcal{H}'_{1,8}{}^{(2)}} + \sum_{H' \in f_2(\mathcal{H}'_{1,8}{}^{(2)})} \right) (W(H) - W(H')) = 0.$$

Subcase 1.9: $uw, ww', uv, vv' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 1) \cdot 1 \cdot 1 \cdot 1 \cdot N$, for some constant value N . $W(H) = (b_1 + 1) \cdot (b_2 + 1) \cdot (b_3 + 1) \cdot 1 \cdot 1 \cdot N$, so $W(H) - W(H') = [b_1 \cdot b_2 \cdot b_3 + b_1 \cdot b_2 + b_1 \cdot b_3 + b_2 \cdot b_3] \cdot N \geq 0$. Denote $\mathcal{H}'_{1,9} = \{H' | u \in T'_1, uw, ww', uv, vv' \notin H'\}$.

Subcase 1.10: $ww' \in H', uw, uv, vv' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 1) \cdot 1 \cdot 1 \cdot 2 \cdot N$, for some constant value N . $W(H) = (b_1 + 1) \cdot (b_2 + 2) \cdot (b_3 + 1) \cdot 1 \cdot N$, so $W(H) - W(H') = [b_2 \cdot (b_1 \cdot b_3 + b_1 + b_3 - 1) + 2 \cdot b_1 \cdot b_3] \cdot N$. Denote $\mathcal{H}'_{1,10}{}^{(1)} = \{H' | u \in T'_1, ww' \in H', uw, uv, vv' \notin H', b_1 + b_3 \neq 0\}$. If $H' \in \mathcal{H}'_{1,9}{}^{(1)}$, $W(H) - W(H') \geq 0$. Denote $\mathcal{H}'_{1,10}{}^{(2)} = \{H' | u \in T'_1, ww' \in H', uw, uv, vv' \notin H', b_1 = b_3 = 0\}$. For every $H'_1 \in \mathcal{H}'_{1,10}{}^{(2)}$, set $H'_2 = H'_1 - ww' + uv$, it is obvious that $H'_2 \in \mathcal{H}'_{1,1}$ in which $b_1 = 0, b_3 = 0$ and $f_3 : \mathcal{H}'_{1,10}{}^{(2)} \rightarrow \mathcal{H}'_{1,1}$ is an injection. Then

$$\left(\sum_{H' \in \mathcal{H}'_{1,10}{}^{(2)}} + \sum_{H' \in \mathcal{H}'_{1,1}} \right) (W(H) - W(H')) \geq \left(\sum_{H' \in \mathcal{H}'_{1,10}{}^{(2)}} + \sum_{H' \in f_3(\mathcal{H}'_{1,10}{}^{(2)})} \right) (W(H) - W(H')) = 0.$$

Subcase 1.11: $vv' \in H', uw, uv, ww' \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 1) \cdot 1 \cdot 1 \cdot 2 \cdot N$, for some constant value N . $W(H) = (b_1 + 1) \cdot (b_2 + 1) \cdot (b_3 + 2) \cdot 1 \cdot N$, so $W(H) - W(H') = [b_3 \cdot (b_1 \cdot b_2 + b_1 + b_2 - 1) + 2 \cdot b_1 \cdot b_2] \cdot N$. Denote $\mathcal{H}'_{1,11}{}^{(1)} = \{H' | u \in T'_1, vv' \in H', uw, uv, ww' \notin H', b_1 + b_2 \neq 0\}$. If $H' \in \mathcal{H}'_{1,11}{}^{(1)}$, $W(H) - W(H') \geq 0$. Denote $\mathcal{H}'_{1,11}{}^{(2)} = \{H' | u \in T'_1, vv' \in H', uw, uv, ww' \notin H', b_1 = b_2 = 0\}$. For every $H'_1 \in \mathcal{H}'_{1,11}{}^{(2)}$, assume u_2 is in a component of H'_1 of order p . Set $H'_2 = H'_1 - vv' + uu_2$, it is obvious that $H'_2 \in \mathcal{H}'_{1,9}$ in which $b_1 = p \geq 1, b_2 = 0$ and $f_4 : \mathcal{H}'_{1,11}{}^{(2)} \rightarrow \mathcal{H}'_{1,9}$ is an injection. Then

$$\left(\sum_{H' \in \mathcal{H}'_{1,11}{}^{(2)}} + \sum_{H' \in \mathcal{H}'_{1,9}} \right) (W(H) - W(H')) \geq \left(\sum_{H' \in \mathcal{H}'_{1,11}{}^{(2)}} + \sum_{H' \in f_4(\mathcal{H}'_{1,11}{}^{(2)})} \right) (W(H) - W(H')) = 0.$$

Subcase 1.12: $vv', ww' \in H', uw, uv \notin H'$, then $W(H') = (b_1 + b_2 + b_3 + 1) \cdot 2 \cdot 2 \cdot N$, for some constant value N . $W(H) = (b_1 + 1) \cdot (b_2 + 2) \cdot (b_3 + 2) \cdot N$, so $W(H) - W(H') = [b_1 \cdot b_2 \cdot b_3 + b_2 \cdot b_3 + 2 \cdot (b_1 - 1) \cdot (b_2 + b_3)] \cdot N$. Denote $\mathcal{H}'_{1,12}{}^{(1)} = \{H' | u \in T'_1, vv', ww' \in H', uw, uv \notin H', b_1 \geq 1 \text{ or } b_1 = b_2 = b_3 = 0\}$. If $H' \in \mathcal{H}'_{1,12}{}^{(1)}$, $W(H) - W(H') \geq 0$.

Denote $\mathcal{H}_{1,12}'^{(2)} = \{H' | u \in T'_1, vv', ww' \in H', uw, uv \notin H', b_1 = 0, b_3 = 0, b_2 \geq 1\}$. For every $H'_1 \in \mathcal{H}_{1,12}'^{(2)}$, set $H'_2 = H'_1 - ww' + uv$, it is obvious that $H'_2 \in \mathcal{H}_{1,2}'$ in which $b_1 = 0, b_3 = 0$ and $f_5 : \mathcal{H}_{1,12}'^{(2)} \rightarrow \mathcal{H}_{1,2}'$ is an injection. Then

$$(\sum_{H' \in \mathcal{H}_{1,12}'^{(2)}} + \sum_{H' \in \mathcal{H}_{1,2}'})(W(H) - W(H')) \geq (\sum_{H' \in \mathcal{H}_{1,12}'^{(2)}} + \sum_{H' \in f_5(\mathcal{H}_{1,12}'^{(2)})})(W(H) - W(H')) = 0.$$

Denote $\mathcal{H}_{1,12}'^{(3)} = \{H' | u \in T'_1, vv', ww' \in H', uw, uv \notin H', b_1 = 0, b_3 \geq 1\}$. For every $H'_1 \in \mathcal{H}_{1,12}'^{(3)}$, assume u_2 is in a component of H'_1 of order p . Set $H'_2 = H'_1 - vv' + uu_2$, it is obvious that $H'_2 \in \mathcal{H}_{1,10}'^{(1)}$ in which $b_1 = p \geq 1, b_3 \geq 1$ and denote this kind of subset of $\mathcal{H}_{1,10}'^{(1)}$ as $\mathcal{H}_{1,10}'^{(11)}$. Moreover, $f_6 : \mathcal{H}_{1,12}'^{(3)} \rightarrow \mathcal{H}_{1,10}'^{(11)}$ is an injection. Then

$$(\sum_{H' \in \mathcal{H}_{1,12}'^{(3)}} + \sum_{H' \in \mathcal{H}_{1,10}'^{(11)}})(W(H) - W(H')) \geq (\sum_{H' \in \mathcal{H}_{1,12}'^{(3)}} + \sum_{H' \in f_6(\mathcal{H}_{1,12}'^{(3)})})(W(H) - W(H')) \geq 0.$$

Case 2: u is in an odd unicyclic component U' of H' .

Subcase 2.1: There is exactly one edge among $\{uv, uw\}$ which belongs to $E(H')$. Assume $|V(U') \setminus \{v, v', w, w'\}| = x$, where $x \geq g(G') \geq 3$. If $vv', ww' \notin H'$, then it is obvious that there is a bijection between the two sets $\mathcal{H}'_{2,1} = \{H' | u \in U', uw \in H', vv', ww', uv \notin H'\}$, $\mathcal{H}'_{2,2} = \{H' | u \in U', uv \in H', vv', ww', uw \notin H'\}$. Then

$$\sum_{H' \in \mathcal{H}'_{2,1} \cup \mathcal{H}'_{2,2}} [W(H) - W(H')] = \sum_{H' \in \mathcal{H}'_{2,1} \cup \mathcal{H}'_{2,2}} (2 \cdot x + 4 - 8) \cdot N \geq 0.$$

If there is exactly one edge among $\{vv', ww'\}$ which belongs to $E(H')$, then it is obvious that there is a bijection between every two sets of $\mathcal{H}'_{2,3} = \{H' | u \in U', uv, vv' \in H', ww', uw \notin H'\}$, $\mathcal{H}'_{2,4} = \{H' | u \in U', uv, ww' \in H', vv', uw \notin H'\}$, $\mathcal{H}'_{2,5} = \{H' | u \in U', uw, vv' \in H', ww', uv \notin H'\}$, $\mathcal{H}'_{2,6} = \{H' | u \in U', uw, ww' \in H', vv', uv \notin H'\}$. Then

$$\sum_{H' \in \mathcal{H}'_{2,3} \cup \mathcal{H}'_{2,4} \cup \mathcal{H}'_{2,5} \cup \mathcal{H}'_{2,6}} [W(H) - W(H')] = \sum_{H' \in \mathcal{H}'_{2,3} \cup \mathcal{H}'_{2,4} \cup \mathcal{H}'_{2,5} \cup \mathcal{H}'_{2,6}} (4 \cdot x + 12 - 24) \cdot N \geq 0.$$

If $vv', ww' \in H'$, then it is obvious that there is a bijection between the two sets $\mathcal{H}'_{2,7} = \{H' | u \in U', uv, vv', ww' \in H', uw \notin H'\}$, $\mathcal{H}'_{2,8} = \{H' | u \in U', uw, vv', ww' \in H', uv \notin H'\}$. Then

$$\sum_{H' \in \mathcal{H}'_{2,7} \cup \mathcal{H}'_{2,8}} [W(H) - W(H')] = \sum_{H' \in \mathcal{H}'_{2,7} \cup \mathcal{H}'_{2,8}} (2 \cdot x + 8 - 16) \cdot N.$$

When $x \geq 4$, the above equation is nonnegative. When $x = 3$, then $g(G') = 3$, since $uu' \in E(G)$, for every $H'_1 \in \mathcal{H}'_{2,7}, H'_2 \in \mathcal{H}'_{2,8}$, set $H'_3 = H'_1 - u_1w_1 + uu', H'_4 = H'_2 - u_1w_1 + uu'$. It is easy to obtain that $H'_3 \in \mathcal{H}'_{1,4}, H'_4 \in \mathcal{H}'^{(1)}_{1,8}$, in which $b_1 = 2, b_2 = 1, b_3 = 0, g(G') = 3$ and denote this kind of subset of $\mathcal{H}'_{1,8}$ as $\mathcal{H}'^{(11)}_{1,8}$. Moreover, $f_7 : \mathcal{H}'_{2,7} \rightarrow \mathcal{H}'_{1,4}$ and $f_8 : \mathcal{H}'_{2,8} \rightarrow \mathcal{H}'^{(11)}_{1,8}$ are injections. Then

$$\sum_{H' \in \mathcal{H}'_{2,7} \cup \mathcal{H}'_{2,8} \cup \mathcal{H}'_{1,4} \cup \mathcal{H}'^{(11)}_{1,8}} [W(H) - W(H')] \geq \sum_{H' \in \mathcal{H}'_{2,7} \cup \mathcal{H}'_{2,8} \cup f_7(\mathcal{H}'_{2,7}) \cup f_8(\mathcal{H}'_{2,8})} (-2 + 2 + 3) > 0.$$

Subcase 2.2: When $uw, uv \notin H'$, assume $|V(U') \setminus (V(T_v^G) \cup \{u\})| = b$ and $|V(U') \cap V(T_v^G)| = b_3$, then $|V(U')| = b + b_3 + 1$, with $b \geq g(G') - 1 \geq 2, b_3 \geq 0$. Denote N the product of the orders of all components of H' except the components containing $\{u, v, w, v', w'\}$.

If $vv', ww' \notin H'$, then $W(H) - W(H') = [(b_3 + 1) \cdot (b + 2) - 4] \cdot N \geq 2 \cdot b_3 \cdot N \geq 0$. Denote $\mathcal{H}'_{2,9} = \{H' | u \in U', uw, uv, vv', ww' \notin H'\}$.

If there is exactly one edge in $\{vv', ww'\}$ which belongs to $E(H')$, then it is obvious that there is a bijection between the two sets $\mathcal{H}'_{2,10} = \{H' | u \in U', ww' \in H', vv', uv, uw \notin H'\}, \mathcal{H}'_{2,11} = \{H' | u \in U', vv' \in H', ww', uv, uw \notin H'\}$. Then

$$\begin{aligned} \sum_{H' \in \mathcal{H}'_{2,10} \cup \mathcal{H}'_{2,11}} [W(H) - W(H')] &= \sum_{H' \in \mathcal{H}'_{2,10} \cup \mathcal{H}'_{2,11}} ((b_3 + 2) \cdot (b + 2) + (b_3 + 1) \cdot (b + 3) - 16) \cdot N \\ &\geq \sum_{H' \in \mathcal{H}'_{2,10} \cup \mathcal{H}'_{2,11}} (2 \cdot b_3 \cdot b + 3 \cdot b + 5 \cdot b_3 - 9) \cdot N. \end{aligned}$$

When $b \geq 3$ or $b = 2, b_3 \geq 1$, the above equation is nonnegative, and denote this kind of subset of $\mathcal{H}'_{2,10}, \mathcal{H}'_{2,11}$ as $\mathcal{H}'^{(1)}_{2,10}, \mathcal{H}'^{(1)}_{2,11}$, respectively. Denote

$$\mathcal{H}'^{(2)}_{2,10} = \{H' | u \in U', ww' \in H', vv', uv, uw \notin H', b = 2, b_3 = 0\},$$

$$\mathcal{H}'^{(2)}_{2,11} = \{H' | u \in U', vv' \in H', ww', uv, uw \notin H', b = 2, b_3 = 0\}.$$

When $b = 2, b_3 = 0$, then $g(G') = 3$ and $|V(U')| = 3$, since the pendent edge $uu' \in E(G)$, for every $H'_1 \in \mathcal{H}'^{(2)}_{2,10}, H'_2 \in \mathcal{H}'^{(2)}_{2,11}$, set $H'_3 = H'_1 - u_2w_2 + uu', H'_4 = H'_2 - u_2w_2 + uu'$. It is obvious that $H'_3 \in \mathcal{H}'^{(1)}_{1,10}$, in which $b_1 = 2, b_2 = 1, b_3 = 0$, and denote this kind of subset of $\mathcal{H}'^{(1)}_{1,10}$ as $\mathcal{H}'^{(12)}_{1,10}$. $H'_4 \in \mathcal{H}'^{(1)}_{1,11}$, in which $b_1 = 2, b_2 = 1, b_3 = 0$, and denote this kind of subset of $\mathcal{H}'^{(1)}_{1,11}$ as $\mathcal{H}'^{(11)}_{1,11}$. Moreover, $f_9 : \mathcal{H}'^{(2)}_{2,10} \rightarrow \mathcal{H}'^{(12)}_{1,10}$ is an injection, $f_{10} : \mathcal{H}'^{(2)}_{2,11} \rightarrow \mathcal{H}'^{(11)}_{1,11}$ is an injection. Then

$$\sum_{H' \in \mathcal{H}'^{(2)}_{2,10} \cup \mathcal{H}'^{(2)}_{2,11} \cup \mathcal{H}'^{(12)}_{1,10} \cup \mathcal{H}'^{(11)}_{1,11}} [W(H) - W(H')]$$

$$\geq \sum_{H' \in \mathcal{H}'_{2,10}(2) \cup \mathcal{H}'_{2,11}(2) \cup f_9(\mathcal{H}'_{2,10}(2)) \cup f_{10}(\mathcal{H}'_{2,11}(2))} (-3 + 4 + 1) \cdot N > 0.$$

If $vv', ww' \in H'$, then $W(H) - W(H') = [(b_3 + 2) \cdot (b + 3) - 16] \cdot N = [b_3 \cdot b + 3 \cdot b_3 + 2 \cdot b - 10] \cdot N$. When $b = 2, b_3 \geq 2$ or $b = 3, b_3 \geq 1$ or $b = 4, b_3 \geq 1$ or $b \geq 5$, the above equation is nonnegative.

When $b = 2, b_3 = 0$, then $g(G') = 3$ and $|V(U')| = 3$. Denote $\mathcal{H}'_{2,12}(1) = \{H' | u \in U', vv', ww' \in H', uv, uw \notin H', b = 2, b_3 = 0\}$. Since $n \geq 9$, without loss of generality, assume $uu_0 \in E(G)$, for every $H'_1 \in \mathcal{H}'_{2,12}(1)$, set $H'_2 = H'_1 - ww' + uu' - u_2w_2 + uu_0$. It is obvious that $H'_2 \in \mathcal{H}'_{1,11}(1)$ in which $b = 2, b_3 = 0$, and denote this kind of subset of $\mathcal{H}'_{1,11}(1)$ as $\mathcal{H}'_{1,11}(12)$. Moreover, $f_{11} : \mathcal{H}'_{2,12}(1) \rightarrow \mathcal{H}'_{1,11}(12)$ is an injection. Then

$$\sum_{H' \in \mathcal{H}'_{2,12}(1) \cup \mathcal{H}'_{1,11}(12)} [W(H) - W(H')] \geq \sum_{H' \in \mathcal{H}'_{2,12}(1) \cup f_{11}(\mathcal{H}'_{2,12}(1))} (-6 + 6) \cdot N = 0.$$

When $b = 2, b_3 = 1$, then $g(G') = 3$ and $|V(U')| = 4$. Denote $\mathcal{H}'_{2,12}(2) = \{H' | u \in U', vv', ww' \in H', uv, uw \notin H', b = 2, b_3 = 1\}$. For every $H'_1 \in \mathcal{H}'_{2,12}(2)$, set $H'_2 = H'_1 - uw_2 + uw$. It is obvious that $H'_2 \in \mathcal{H}'_{1,8}(1)$ in which $b_1 = 2, b_2 = 0, b_3 = 1$, and denote this kind of subset of $\mathcal{H}'_{1,8}(1)$ as $\mathcal{H}'_{1,8}(12)$. Moreover, $f_{12} : \mathcal{H}'_{2,12}(2) \rightarrow \mathcal{H}'_{1,8}(12)$ is an injection.

$$\sum_{H' \in \mathcal{H}'_{2,12}(2) \cup \mathcal{H}'_{1,8}(12)} [W(H) - W(H')] \geq \sum_{H' \in \mathcal{H}'_{2,12}(2) \cup f_{12}(\mathcal{H}'_{2,12}(2))} (-1 + 3) \cdot N > 0.$$

When $b = 3, b_3 = 0$, then $g(G') = 3$ and $|V(U')| = 4$. Denote $\mathcal{H}'_{2,12}(3) = \{H' | u \in U', vv', ww' \in H', uv, uw \notin H', b = 3, b_3 = 0\}$. For every $H'_1 \in \mathcal{H}'_{2,12}(3)$, set $H'_2 = H'_1 - uw_2 + uw$. It is obvious that $H'_2 \in \mathcal{H}'_{1,8}(1)$ in which $b_1 = 3, b_2 = 0, b_3 = 0$, and denote this kind of subset of $\mathcal{H}'_{1,8}(1)$ as $\mathcal{H}'_{1,8}(13)$. Moreover, $f_{13} : \mathcal{H}'_{2,12}(3) \rightarrow \mathcal{H}'_{1,8}(13)$ is an injection.

$$\sum_{H' \in \mathcal{H}'_{2,12}(3) \cup \mathcal{H}'_{1,8}(13)} [W(H) - W(H')] \geq \sum_{H' \in \mathcal{H}'_{2,12}(3) \cup f_{13}(\mathcal{H}'_{2,12}(3))} (-4 + 4) \cdot N = 0.$$

When $b = 4, b_3 = 0$, then $|V(U')| = 5$ and $g(G') = 3$ or $g(G') = 5$. Denote $\mathcal{H}'_{2,12}(4) = \{H' | u \in U', vv', ww' \in H', uv, uw \notin H', b = 4, b_3 = 0, g(G') = 3\}$, $\mathcal{H}'_{2,12}(5) = \{H' | u \in U', vv', ww' \in H', uv, uw \notin H', b = 4, b_3 = 0, g(G') = 5\}$. For every $H'_1 \in \mathcal{H}'_{2,12}(4), H'_2 \in \mathcal{H}'_{2,12}(5)$, set $H'_3 = H'_1 - uw_2 + uw, H'_4 = H'_2 - uw_2 + uw$. It is obvious that

$H'_3 \in \mathcal{H}'_{1,8}{}^{(1)}$ in which $b_1 = 4, b_2 = 0, b_3 = 0, g(G') = 3$, and denote this kind of subset of $\mathcal{H}'_{1,8}{}^{(1)}$ as $\mathcal{H}'_{1,8}{}^{(14)}$. Moreover, $f_{14} : \mathcal{H}'_{2,12}{}^{(3)} \rightarrow \mathcal{H}'_{1,8}{}^{(14)}$ is an injection.

$$\sum_{H' \in \mathcal{H}'_{2,12}{}^{(4)} \cup \mathcal{H}'_{1,8}{}^{(14)}} [W(H) - W(H')] \geq \sum_{H' \in \mathcal{H}'_{2,12}{}^{(4)} \cup f_{14}(\mathcal{H}'_{2,12}{}^{(4)})} (-2 + 6) \cdot N > 0.$$

Furthermore, $H'_4 \in \mathcal{H}'_{1,8}{}^{(1)}$ in which $b_1 = 4, b_2 = 0, b_3 = 0, g(G') = 5$, and denote this kind of subset of $\mathcal{H}'_{1,8}{}^{(1)}$ as $\mathcal{H}'_{1,8}{}^{(15)}$. Moreover, $f_{15} : \mathcal{H}'_{2,12}{}^{(5)} \rightarrow \mathcal{H}'_{1,8}{}^{(15)}$ is an injection.

$$\sum_{H' \in \mathcal{H}'_{2,12}{}^{(5)} \cup \mathcal{H}'_{1,8}{}^{(15)}} [W(H) - W(H')] \geq \sum_{H' \in \mathcal{H}'_{2,12}{}^{(5)} \cup f_{15}(\mathcal{H}'_{2,12}{}^{(5)})} (-2 + 6) \cdot N > 0.$$

Thus by summing over all possible subsets of \mathcal{H}'_i , from Theorem 1.2 and f is an injection on the whole. Then

$$\varphi_i(G') = \sum_{H' \in \mathcal{H}'_i} W(H') < \sum_{H \in \mathcal{H}_i^*} W(H) \leq \sum_{H \in \mathcal{H}_i} W(H) = \varphi_i(G)$$

holds for $i = 2, 3, \dots, n-1$.

When g is even, by Case 1, $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n-1$ holds. ■

Remark. Assume there are t_i (resp. t_{i+1}, t_{i+2}) pendent edges and s_i (resp. s_{i+1}, s_{i+2}) pendent paths of length 2 attached to u (resp. v, w).

If $t_i, t_{i+1}, t_{i+2} \neq 0$, without loss of generality, assume $uu'', vv'', ww'' \in M(G)$, where u'', v'', w'' are pendent vertices attached at u, v, w . Then $M(G') = M(G) - \{uu'', vv'', ww''\} + \{uu', vv', ww'\}$, we have $M(G') = M(G)$.

3 The ordering of graphs in the two sets $\mathcal{G}_3(s_1, t_1; s_2, t_2; s_3, t_3)$ and $\mathcal{G}_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)$

Lemma 3.1 *For an arbitrary graph*

$$G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4) \in \mathcal{G}_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4),$$

after removing all pendent edges or pendent paths of lengths 2 at u_2, u_3, u_4 , we obtain $G_4(\sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i; 0, 0; 0, 0; 0, 0)$. Then

$$\varphi_i(G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)) \geq \varphi_i(G_4(\sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i; 0, 0; 0, 0; 0, 0)),$$

with equality if and only if $i \in \{0, 1, n-1, n\}$. (see fig.5).

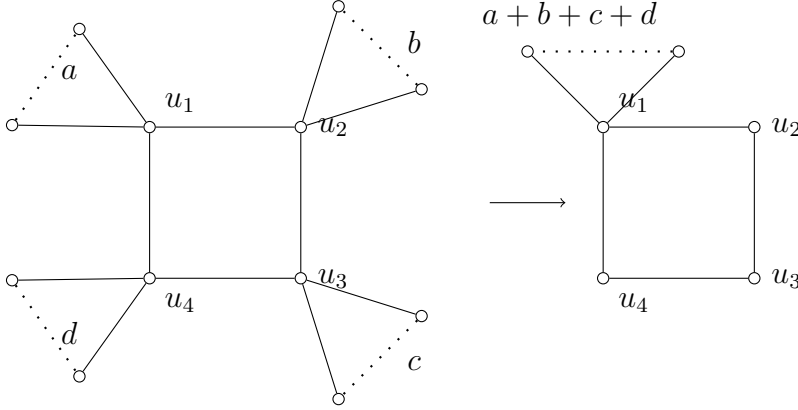


Figure 5: Transformation of Lemma 3.1

Proof. For convenience, we denote the graph $G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)$ as G , and $G_4(\sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i; 0, 0; 0, 0; 0, 0)$ as G' .

When $i \in \{0, 1, n-1, n\}$, the proof is similar to Theorem 2.2. For $2 \leq i \leq n-2$, we use \mathcal{H}'_i and \mathcal{H}_i to denote the set of all TU-subgraphs of G and G' with exactly i edges, respectively. Let $\mathcal{H}'_i = \mathcal{H}'^{(1)}_i \cup \mathcal{H}'^{(2)}_i \cup \mathcal{H}'^{(3)}_i \cup \mathcal{H}'^{(4)}_i$, where $\mathcal{H}'^{(j)}_i (j = 1, 2, 3, 4)$ is the set of TU-subgraphs in which u_1, u_2, u_3, u_4 belong to exactly j different components. Similarly, we can define $\mathcal{H}^{(j)}_i (j = 1, 2, 3, 4)$.

For an arbitrary TU-subgraph $H' \in \mathcal{H}'^{(j)}_i$, let R' be the component of H' containing u_1 . Let $N_{R'}(u_1) \cap N_{T_{u_1}^G}(u_1) = \{u_1^1, u_1^2, \dots, u_1^a\}$, where $0 \leq a \leq \min\{d_G(u_1) - 2, |V(R')| - 1\}$, $N_{R'}(u_1) \cap N_{T_{u_2}^G}(u_2) = \{u_2^1, u_2^2, \dots, u_2^b\}$, where $0 \leq b \leq \min\{d_G(u_2) - 2, |V(R')| - 1\}$, $N_{R'}(u_1) \cap N_{T_{u_3}^G}(u_3) = \{u_3^1, u_3^2, \dots, u_3^c\}$, where $0 \leq c \leq \min\{d_G(u_3) - 2, |V(R')| - 1\}$, $N_{R'}(u_1) \cap N_{T_{u_4}^G}(u_4) = \{u_4^1, u_4^2, \dots, u_4^d\}$, where $0 \leq d \leq \min\{d_G(u_4) - 2, |V(R')| - 1\}$. For G , define H with $V(H) = V(H')$ and

$$\begin{aligned} E(H) = & E(H') - u_1u_2^1 - \dots - u_1u_2^b - u_1u_3^1 - \dots - u_1u_3^c - u_1u_4^1 - \dots - u_1u_4^d \\ & + u_2u_2^1 + \dots + u_2u_2^b + u_3u_3^1 + \dots + u_3u_3^c + u_4u_4^1 + \dots + u_4u_4^d. \end{aligned}$$

Then $H \in \mathcal{H}_i$. Obviously, $H' \in \mathcal{H}'^{(j)}_i \Leftrightarrow H \in \mathcal{H}^{(j)}_i, (j = 1, 2, 3, 4)$. Let $f : \mathcal{H}'^{(j)}_i \rightarrow \mathcal{H}^{(j)}_i$, and $\mathcal{H}^{*(j)}_i = f(\mathcal{H}'^{(j)}_i) = \{f(H') | H' \in \mathcal{H}'^{(j)}_i\}$.

Denote $|V(T_{u_1}^G) \cap V(R') \setminus \{u_1\}| = A, |V(T_{u_2}^G) \cap V(R') \setminus \{u_2\}| = B, |V(T_{u_3}^G) \cap V(R') \setminus \{u_3\}| = C, |V(T_{u_4}^G) \cap V(R') \setminus \{u_4\}| = D$, where $A, B, C, D \geq 0$.

Now we distinguish the proof into four cases.

Case 1: $H' \in \mathcal{H}'^{(1)}_i$, u_1, u_2, u_3, u_4 belong to one component, then $W(H) = W(H')$,

thus

$$\sum_{H' \in \mathcal{H}_i'^{(1)}} [W(H) - W(H')] = 0.$$

Case 2: $H' \in \mathcal{H}_i'^{(4)}$, u_1, u_2, u_3, u_4 are in four trees, then

$$W(H) - W(H') = (A+1)(B+1)(C+1)(D+1)N_1 - (A+B+C+D+1)N_1 \geq 0,$$

for some constant value N_1 . Thus

$$\sum_{H' \in \mathcal{H}_i'^{(4)}} [W(H) - W(H')] \geq 0.$$

Case 3: $H' \in \mathcal{H}_i'^{(2)}$, u_1, u_2, u_3, u_4 are in two trees, then by computing,

$$\begin{aligned} W(H) - W(H') &= [(A+D+2)(B+C+2) - 2(A+B+C+D+2)]N_2 \\ &\quad + [(A+B+2)(C+D+2) - 2(A+B+C+D+2)]N_2 \\ &\quad + [AD+BD+CD+AC+BC+CD+AB+BD+BC+AB+AC+AD]N_2 \geq 0, \end{aligned}$$

for some constant value N_2 . Thus

$$\sum_{H' \in \mathcal{H}_i'^{(2)}} [W(H) - W(H')] \geq 0.$$

Case 4: $H' \in \mathcal{H}_i'^{(3)}$, u_1, u_2, u_3, u_4 are in three trees, then

$$\begin{aligned} W(H) - W(H') &= [(AB+A+B)(C+D) + 2AB + (AD+A+D)(B+C) + 2AD \\ &\quad + (CD+C+D)(A+B) + 2CD + (BC+B+C)(A+D) + 2BC]N_3 \geq 0, \end{aligned}$$

for some constant value N_3 . Thus

$$\sum_{H' \in \mathcal{H}_i'^{(3)}} [W(H) - W(H')] \geq 0.$$

Now the inequality $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n-2$ holds from Theorem 1.2 by summing over all possible subsets \mathcal{H}_i' of TU-subgraphs H' of G_1 with i edges. ■

Remark. If there is only one positive number in the set $\{t_1, t_2, t_3, t_4\}$, then after the transformation in Lemma 3.1, $M(G) = M(G')$.

If $t_1 = t_3 = 0, t_2, t_4 \neq 0$ or $t_2 = t_4 = 0, t_1, t_3 \neq 0$, then after the transformation in Lemma 3.1, $M(G) = M(G')$.

Similar to the prove of Lemma 3.1, we have the the following Lemma.

Lemma 3.2 *For an arbitrary graph*

$$G_3(s_1, t_1; s_2, t_2; s_3, t_3) \in \mathcal{G}_3(s_1, t_1; s_2, t_2; s_3, t_3),$$

after removing all pendent edges or pendent paths of lengths 2 at u_2, u_3 , we obtain $G_3(\sum_{i=1}^3 s_i, \sum_{i=1}^3 t_i; 0, 0; 0, 0)$. Then

$$\varphi_i(G_3(s_1, t_1; s_2, t_2; s_3, t_3)) \geq \varphi_i(G_3(\sum_{i=1}^3 s_i, \sum_{i=1}^3 t_i; 0, 0; 0, 0)),$$

with equality if and only if $i \in \{0, 1, n-1, n\}$.

Remark. If there is at least one number in $\{t_1, t_2, t_3\}$ which equals to zero, then $M(G_3(s_1, t_1; s_2, t_2; s_3, t_3)) = M(G_3(\sum_{i=1}^3 s_i, \sum_{i=1}^3 t_i; 0, 0; 0, 0))$.

Lemma 3.3 *For a graph*

$$G_3(s_1, t_1; s_2, t_2; s_3, t_3) \in \mathcal{G}_3(s_1, t_1; s_2, t_2; s_3, t_3),$$

which satisfies $t_1 \neq 0, t_2 \neq 0, t_3 \neq 0$, without loss of generality, assume u_1u', u_2v', u_3w' are pendent edges of $G_3(s_1, t_1; s_2, t_2; s_3, t_3)$. After removing all pendent edges or pendent paths of lengths 2 except u_1u', u_2v' from vertices u_1, u_2 to vertex u_3 , we obtain $G_3(0, 1; 0, 1; \sum_{i=1}^3 s_i, \sum_{i=1}^3 t_i - 2)$. Then

$$\varphi_i(G_3(s_1, t_1; s_2, t_2; s_3, t_3)) \geq \varphi_i(G_3(0, 1; 0, 1; \sum_{i=1}^3 s_i, \sum_{i=1}^3 t_i - 2)),$$

with equality if and only if $i \in \{0, 1, n-1, n\}$.

Proof. For convenience, we denote the graph $G_3(s_1, t_1; s_2, t_2; s_3, t_3)$ as G , and denote $G_3(0, 1; 0, 1; \sum_{i=1}^3 s_i, \sum_{i=1}^3 t_i - 2)$ as G' .

When $i \in \{0, 1, n-1, n\}$, the proof is similar to Theorem 2.2. For $2 \leq i \leq n-2$, we use \mathcal{H}'_i and \mathcal{H}_i to denote the set of all TU-subgraphs of G and G' with exactly i edges, respectively. Let $\mathcal{H}'_i = \mathcal{H}'^{(1)}_i \cup \mathcal{H}'^{(2)}_i \cup \mathcal{H}'^{(3)}_i$, where $\mathcal{H}'^{(j)}_i (j = 1, 2, 3)$ is the set of TU-subgraphs in which u_1, u_2, u_3 belong to exactly j different components. Similarly, we can define $\mathcal{H}_i^{(j)} (j = 1, 2, 3)$.

For an arbitrary TU-subgraph $H' \in \mathcal{H}'^{(j)}_i$, let R' be the component of H' containing u_3 . Let $N_{R'}(u_3) \cap N_{T_{u_1}^G}(u_1) = \{u_1, u_1^1, u_1^2, \dots, u_1^a\}$, where $0 \leq a \leq \min\{d_G(u_1) - 2, |V(R')| - 1\}$, $N_{R'}(u_3) \cap N_{T_{u_2}^G}(u_2) = \{u_2, u_2^1, u_2^2, \dots, u_2^b\}$, where $0 \leq b \leq \min\{d_G(u_2) -$

$2, |V(R')| - 1\}$, $N_{R'}(u_3) \cap N_{T_{u_3}^G}(u_3) = \{u_3^1, u_3^2, \dots, u_3^c\}$, where $0 \leq c \leq \min\{d_G(u_3) - 2, |V(R')| - 1\}$. For G , define H with $V(H) = V(H')$ and

$$E(H) = E(H') - u_3u_1^1 - \dots - u_3u_1^a - u_3u_2^1 - \dots - u_3u_2^b \\ + u_1u_1^1 + \dots + u_1u_1^a + u_2u_2^1 + \dots + u_2u_2^b.$$

Then $H \in \mathcal{H}_i$. Obviously, $H' \in \mathcal{H}_i'^{(j)} \Leftrightarrow H \in \mathcal{H}_i^{(j)}$, ($j = 1, 2, 3$). Let $f : \mathcal{H}_i'^{(j)} \rightarrow \mathcal{H}_i^{(j)}$, and $\mathcal{H}_i^{*(j)} = f(\mathcal{H}_i'^{(j)}) = \{f(H') | H' \in \mathcal{H}_i'^{(j)}\}$.

Denote $|V(T_{u_1}^G) \cap V(R') \setminus \{u_1, u'\}| = A$, $|V(T_{u_2}^G) \cap V(R') \setminus \{u_2, v'\}| = B$, $|V(T_{u_3}^G) \cap V(R') \setminus \{u_3, w'\}| = C$, where $A, B, C \geq 0$. Denote N be the product of the orders of all components of H' excluding $u_1, u', u_2, v', u_3, w'$.

Now we distinguish the proof into three cases.

Case 1: $H' \in \mathcal{H}_i'^{(1)}$, u_1, u_2, u_3 belong to one component, then $W(H) = W(H')$, thus

$$\sum_{H' \in \mathcal{H}_i'^{(1)}} [W(H) - W(H')] = 0.$$

Case 2: $H' \in \mathcal{H}_i'^{(2)}$, u_1, u_2, u_3 are in two trees, we distinguish this case into the following four cases.

Subcase 2.1: $|\{u_1u', u_2v', u_3w'\} \cap E(H')| = 1$, and N is fixed.

Denote this kind of subset of $\mathcal{H}_i'^{(2)}$ as $\mathcal{H}_i'^{(21)}$.

(1). Assume $u_1u' \in E(H')$.

(1.1). $u_1u_3 \in E(H'), u_1u_2, u_2u_3 \notin E(H')$.

$$W(H) - W(H') = (A + C + 3)(B + 1)N - (A + B + C + 3)N.$$

(1.2). $u_1u_2 \in E(H'), u_1u_3, u_2u_3 \notin E(H')$.

$$W(H) - W(H') = (A + B + 3)(C + 1)N - 3(A + B + C + 1)N.$$

(1.3). $u_2u_3 \in E(H'), u_1u_2, u_1u_3 \notin E(H')$.

$$W(H) - W(H') = (B + C + 3)(A + 1)N - 2(A + B + C + 2)N.$$

(2). Assume $u_2v' \in E(H')$. By the symmetry of u_1 and u_2 , the discussion is similar to the above.

(3). Assume $u_3w' \in E(H')$.

(3.1). $u_1u_3 \in E(H'), u_1u_2, u_2u_3 \notin E(H')$.

$$W(H) - W(H') = (A + C + 3)(B + 1)N - (A + B + C + 3)N.$$

$$(3.2). \quad u_1u_2 \in E(H'), u_1u_3, u_2u_3 \notin E(H').$$

$$W(H) - W(H') = (A + B + 2)(C + 2)N - 2(A + B + C + 2)N.$$

$$(3.3). \quad u_2u_3 \in E(H'), u_1u_2, u_1u_3 \notin E(H').$$

$$W(H) - W(H') = (B + C + 3)(A + 1)N - (A + B + C + 3)N.$$

Then

$$\sum_{H' \in \mathcal{H}_i'^{(21)}} [W(H) - W(H')] = 6(AB + BC + AC)N \geq 0.$$

Subcase 2.2: $|\{u_1u', u_2v', u_3w'\} \cap E(H')| = 2$, and N is fixed.

Denote this kind of subset of $\mathcal{H}_i'^{(2)}$ as $\mathcal{H}_i'^{(22)}$.

$$(1). \quad \text{Assume } u_1u', u_2v' \in E(H').$$

$$(1.1). \quad u_1u_3 \in E(H'), u_1u_2, u_2u_3 \notin E(H').$$

$$W(H) - W(H') = (A + C + 3)(B + 2)N - 2(A + B + C + 3)N.$$

$$(1.2). \quad u_1u_2 \in E(H'), u_1u_3, u_2u_3 \notin E(H').$$

$$W(H) - W(H') = (A + B + 4)(C + 1)N - 4(A + B + C + 1)N.$$

$$(1.3). \quad u_2u_3 \in E(H'), u_1u_2, u_1u_3 \notin E(H').$$

$$W(H) - W(H') = (B + C + 3)(A + 2)N - 2(A + B + C + 3)N.$$

$$(2). \quad \text{Assume } u_1u', u_3w' \in E(H').$$

$$(2.1). \quad u_1u_3 \in E(H'), u_1u_2, u_2u_3 \notin E(H').$$

$$W(H) - W(H') = (A + C + 4)(B + 1)N - (A + B + C + 4)N.$$

$$(2.2). \quad u_1u_2 \in E(H'), u_1u_3, u_2u_3 \notin E(H').$$

$$W(H) - W(H') = (A + B + 3)(C + 2)N - 3(A + B + C + 2)N.$$

$$(2.3). \quad u_2u_3 \in E(H'), u_1u_2, u_1u_3 \notin E(H').$$

$$W(H) - W(H') = (B + C + 3)(A + 2)N - 2(A + B + C + 3)N.$$

(3). Assume $u_2v', u_3w' \in E(H')$. By the symmetry of A and B , the discussion is similar to the above.

Then

$$\sum_{H' \in \mathcal{H}_i'^{(22)}} [W(H) - W(H')] = 6(AB + BC + AC)N \geq 0.$$

Subcase 2.3: $|\{u_1u', u_2v', u_3w'\} \cap E(H')| = 3$, and N is fixed.

Denote this kind of subset of $\mathcal{H}_i'^{(2)}$ as $\mathcal{H}_i'^{(23)}$.

(1). $u_1u_3 \in E(H'), u_1u_2, u_2u_3 \notin E(H')$.

$$W(H) - W(H') = (A + C + 4)(B + 2)N - 2(A + B + C + 4)N.$$

(2). $u_1u_2 \in E(H'), u_1u_3, u_2u_3 \notin E(H')$.

$$W(H) - W(H') = (A + B + 4)(C + 2)N - 4(A + B + C + 2)N.$$

(3). $u_2u_3 \in E(H'), u_1u_2, u_1u_3 \notin E(H')$.

$$W(H) - W(H') = (B + C + 4)(A + 2)N - 2(A + B + C + 4)N.$$

Then

$$\sum_{H' \in \mathcal{H}_i'^{(23)}} [W(H) - W(H')] = 2(AB + BC + AC)N \geq 0.$$

Subcase 2.4: $|\{u_1u', u_2v', u_3w'\} \cap E(H')| = 0$, and N is fixed.

Denote this kind of subset of $\mathcal{H}_i'^{(2)}$ as $\mathcal{H}_i'^{(23)}$.

(1). $u_1u_3 \in E(H'), u_1u_2, u_2u_3 \notin E(H')$.

$$W(H) - W(H') = (A + C + 2)(B + 1)N - (A + B + C + 2)N.$$

(2). $u_1u_2 \in E(H'), u_1u_3, u_2u_3 \notin E(H')$.

$$W(H) - W(H') = (A + B + 2)(C + 1)N - 2(A + B + C + 1)N.$$

(3). $u_2u_3 \in E(H'), u_1u_2, u_1u_3 \notin E(H')$.

$$W(H) - W(H') = (B + C + 2)(A + 1)N - (A + B + C + 2)N.$$

Then

$$\sum_{H' \in \mathcal{H}_i'^{(24)}} [W(H) - W(H')] = 2(AB + BC + AC)N \geq 0.$$

After summing all above results in Case 2 we have

$$\sum_{H' \in \mathcal{H}_i'^{(2)}} [W(H) - W(H')] = \sum_{H' \in \mathcal{H}_i'^{(21)} \cup \mathcal{H}_i'^{(22)} \cup \mathcal{H}_i'^{(23)} \cup \mathcal{H}_i'^{(24)}} [W(H) - W(H')] \geq 0.$$

Case 3: $H' \in \mathcal{H}_i'^{(3)}$, u_1, u_2, u_3 are in three trees, we again distinguish the following four cases.

Subcase 3.1: $|\{u_1u', u_2v', u_3w'\} \cap E(H')| = 1$, and N is fixed.

Denote this kind of subset of $\mathcal{H}_i'^{(3)}$ as $\mathcal{H}_i'^{(31)}$.

(1). $u_1u' \in E(H')$.

$$W(H) - W(H') = (A + 2)(B + 1)(C + 1)N - 2(A + B + C + 1)N.$$

(2). $u_2v' \in E(H')$.

$$W(H) - W(H') = (A + 1)(B + 2)(C + 1)N - 2(A + B + C + 1)N.$$

(3). $u_3w' \in E(H')$.

$$W(H) - W(H') = (A + 1)(B + 1)(C + 2)N - (A + B + C + 2)N.$$

Then

$$\sum_{H' \in \mathcal{H}_i'^{(31)}} [W(H) - W(H')] = (3ABC + 4AB + 4BC + 4AC)N \geq 0.$$

Subcase 3.2: $|\{u_1u', u_2v', u_3w'\} \cap E(H')| = 2$, and N is fixed.

Denote this kind of subset of $\mathcal{H}_i'^{(3)}$ as $\mathcal{H}_i'^{(32)}$.

(1). $u_1u', u_3w' \in E(H')$.

$$W(H) - W(H') = (A + 2)(B + 1)(C + 2)N - 2(A + B + C + 2)N.$$

(2). $u_2v', u_3w' \in E(H')$.

$$W(H) - W(H') = (A + 1)(B + 2)(C + 2)N - 2(A + B + C + 2)N.$$

(3). $u_1u', u_2v' \in E(H')$.

$$W(H) - W(H') = (A + 2)(B + 2)(C + 1)N - 4(A + B + C + 1)N.$$

Then

$$\sum_{H' \in \mathcal{H}_i'^{(32)}} [W(H) - W(H')] = (3ABC + 5AB + 5BC + 5AC)N \geq 0.$$

Subcase 3.3: $|\{u_1u', u_2v', u_3w'\} \cap E(H')| = 3$, and N is fixed.

Denote this kind of subset of $\mathcal{H}_i^{(3)}$ as $\mathcal{H}_i'^{(33)}$.

$$W(H) - W(H') = (A + 2)(B + 2)(C + 2)N - 4(A + B + C + 2)N.$$

Then

$$\sum_{H' \in \mathcal{H}_i'^{(33)}} [W(H) - W(H')] = (ABC + 2AB + 2BC + 2AC)N \geq 0.$$

Subcase 3.4: $|\{u_1u', u_2v', u_3w'\} \cap E(H')| = 0$, and N is fixed.

Denote this kind of subset of $\mathcal{H}_i^{(3)}$ as $\mathcal{H}_i'^{(34)}$.

$$W(H) - W(H') = (A + 1)(B + 1)(C + 1)N - (A + B + C + 1)N.$$

Then

$$\sum_{H' \in \mathcal{H}_i'^{(34)}} [W(H) - W(H')] = (ABC + AB + BC + AC)N \geq 0.$$

After summing all above results in Case 3 we have

$$\sum_{H' \in \mathcal{H}_i'^{(3)}} [W(H) - W(H')] = \sum_{H' \in \mathcal{H}_i'^{(31)} \cup \mathcal{H}_i'^{(32)} \cup \mathcal{H}_i'^{(33)} \cup \mathcal{H}_i'^{(34)}} [W(H) - W(H')] \geq 0.$$

By the assumption of this Lemma, there is at least one TU-subgraph H' with $AB \geq 1$ or $BC \geq 1$ or $AC \geq 1$, thus we can get

$$\varphi_i(G) - \varphi_i(G') = \sum_{H \in \mathcal{H}_i} W(H) - \sum_{H' \in \mathcal{H}_i'} W(H') = \sum_{j=1}^3 \left(\sum_{H \in \mathcal{H}_i^{(j)}} W(H) - \sum_{H' \in \mathcal{H}_i'^{(j)}} W(H') \right) > 0.$$

■

Remark. If $t_1 \neq 0, t_2 \neq 0, t_3 \neq 0$, then

$$M(G_3(s_1, t_1; s_2, t_2; s_3, t_3)) = M(G_3(0, 1; 0, 1; \sum_{i=1}^3 s_i, \sum_{i=1}^3 t_i - 2)).$$

Similar to the prove of Lemma 3.3, we have the the following Lemma.

Lemma 3.4 (1). For a graph

$$G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4) \in \mathcal{G}_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4),$$

which satisfies $t_1 \geq 1, t_2 \geq 1, t_3 \geq 1, t_4 \geq 1$, without loss of generality, assume $u_1u'_1, u_2u'_2, u_3u'_3, u_4u'_4$ are pendent edges of $G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)$. After removing all pendent edges or pendent paths of lengths 2 except $u_1u'_1, u_2u'_2, u_3u'_3$ from vertices u_1, u_2, u_3 to vertex u_4 , we obtain $G_4(0, 1; 0, 1; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 3)$. Then

$$\varphi_i(G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)) \geq \varphi_i(G_4(0, 1; 0, 1; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 3)).$$

(2). For a graph

$$G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4) \in \mathcal{G}_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4),$$

which satisfies $t_1 = 0, t_2 \geq 1, t_3 \geq 1, t_4 \geq 1$, without loss of generality, assume $u_2u'_2, u_3u'_3, u_4u'_4$ are pendent edges of $G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)$. After removing all pendent edges or pendent paths of lengths 2 except $u_2u'_2, u_4u'_4$ from vertices u_2, u_4 to vertex u_3 , we obtain $G_4(0, 1; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 2; 0, 1)$. Then

$$\varphi_i(G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)) \geq \varphi_i(G_4(0, 1; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 2; 0, 1)).$$

(3). For a graph

$$G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4) \in \mathcal{G}_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4),$$

which satisfies $t_1 = 0, t_2 = 0, t_3 \geq 1, t_4 \geq 1$, without loss of generality, assume $u_3u'_3, u_4u'_4$ are pendent edges of $G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)$. After removing all pendent edges or pendent paths of lengths 2 except $u_3u'_3$ from vertices u_3 to vertex u_4 , we obtain $G_4(0, 0; 0, 0; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 1)$. Then

$$\varphi_i(G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)) \geq \varphi_i(G_4(0, 0; 0, 0; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 1)).$$

Remark. If $t_1, t_2, t_3, t_4 \geq 1$, then

$$M(G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)) = M(G_4(0, 1; 0, 1; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 3)).$$

If $t_1 = 0, t_2 \geq 1, t_3 \geq 1, t_4 \geq 1$, then

$$M(G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)) = M(G_4(0, 1; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 2; 0, 1)).$$

If $t_1 = t_2 = 0, t_3 \geq 1, t_4 \geq 1$, then

$$M(G_4(s_1, t_1; s_2, t_2; s_3, t_3; s_4, t_4)) = M(G_4(0, 0; 0, 0; 0, 1; \sum_{i=1}^4 s_i, \sum_{i=1}^4 t_i - 1)).$$

For any graph G and $v \in V(G)$, let $Q_{G|v}(x)$ be the principal submatrix of $Q_G(x)$ obtained by deleting the row and column corresponding to the vertex v . Similar to the proof of Theorem 1.5 and Theorem 1.6, we can prove the following three lemmas.

Lemma 3.5 *If $G = G_1|u : G_2|v$, then $Q_G(x) = Q_{G_1}(x)Q_{G_2}(x) - Q_{G_1}(x)Q_{G_2|v}(x) - Q_{G_2}(x)Q_{G_1|u}(x)$.*

Lemma 3.6 *If G be a connected graph with n vertices which consists of a subgraph $H(V(H) \geq 2)$ and $n - |V(H)|$ pendent vertices attached to a vertex v in H , then $Q_G(x) = (x - 1)^{(n-|V(H)|)}Q_H(x) - (n - |V(H)|)x(x - 1)^{(n-|V(H)|-1)}Q_{H|v}(x)$.*

Lemma 3.7 *If G be a connected graph with n vertices which consists of a subgraph $H(V(H) \geq 2)$ and $\frac{n-|V(H)|}{2}$ pendent paths of length 2 attached to a vertex v in H , denote $\frac{n-|V(H)|}{2} = k$, then $Q_G(x) = (x^2 - 3x + 1)^k Q_H(x) - kx(x - 2)(x^2 - 3x + 1)^{k-1} Q_{H|v}(x)$.*

Let $f(x) = \sum_{i=0}^n (-1)^i a_i x^{n-i}$, $g(x) = \sum_{j=0}^m (-1)^j a_j x^{m-j}$, $a_i > 0, b_j > 0$. Then it is easy to see $f(x)g(x) = f(x) = \sum_{k=0}^{m+n} (-1)^k \sum_{i=0}^k a_i b_{k-i} x^{n+m-k}$ has coefficients alternate with positive and negative.

Denote

$$G_3^1 = G_3(0, 0; 0, 0; m - 2, n - 2m + 1),$$

$$G_3^2 = G_3(0, 1; 0, 1; m - 3, n - 2m + 1),$$

and

$$G_4^1 = G_4(0, 0; 0, 0; 0, 0; m - 2, n - 2m),$$

$$G_4^2 = G_4(0, 0; 0, 0; 0, 1; m - 3, n - 2m + 1),$$

$$G_4^3 = G_4(0, 0; 0, 1; m - 3, n - 2m; 0, 1),$$

$$G_4^4 = G_4(0, 1; 0, 1; 0, 1; m - 4, n - 2m + 1).$$

By using Lemma 3.6 and Lemma 3.7, we can compute the signless Laplacian polynomials of several special n -vertex unicyclic graphs with fixed matching number m . For convenience, write $Q_G(x)$ as $Q(G, x)$.

$$Q(G_3^1, x) = (x^2 - 3x + 1)^{(m-3)}(x - 1)^{(n-2m+1)}[x^5 + (-8 - n + m)x^4 + (-6m + 22 + 6n)x^3 + (9m - 25 - 10n)x^2 + (12 + 3n)x - 4],$$

$$Q(G_3^2, x) = (x^2 - 3x + 1)^{(m-3)}(x - 1)^{(n-2m)}[x^6 + (-9 - n + m)x^5 + (-8m + 27 + 8n)x^4 + (18m - 36 - 19n)x^3 + (24 + 14n - 9m)x^2 + (-12 - 3n)x + 4],$$

$$Q(G_4^1, x) = (x^2 - 3x + 1)^{(m-3)}(x - 1)^{(n-2m-1)}x(x - 2)[x^5 + (-8 - n + m)x^4 + (-7m + 22 + 7n)x^3 + (14m - 25 - 15n)x^2 + (10 + 10n - 6m)x - 2n],$$

$$Q(G_4^2, x) = (x^2 - 3x + 1)^{(m-3)}(x - 1)^{(n-2m+1)}x[x^4 + (-8 - n + m)x^3 + (-7m + 20 + 7n)x^2 + (12m - 17 - 13n)x + 4n],$$

$$Q(G_4^3, x) = (x^2 - 3x + 1)^{(m-4)}(x - 1)^{(n-2m-1)}x(x^2 - 4x + 2)[x^6 + (-9 - n + m)x^5 + (27 + 9n - 9m)x^4 + (26m - 33 - 27n)x^3 + (32n - 26m + 14)x^2 + (-14n + 6m)x + 2n],$$

$$Q(G_4^4, x) = (x^2 - 3x + 1)^{(m-5)}(x - 1)^{(n-2m)}x(x^2 - 4x + 2)[x^7 + (-11 - n + m)x^6 + (11n - 11m + 43)x^5 + (-43n + 42m - 74)x^4 + (50 + 74n - 66m)x^3 + (-56n + 38m)x^2 + (-8 + 18n - 6m)x - 2n].$$

Then we have

$$\begin{aligned} & Q(G_3^1, x) - Q(G_3^2, x) \\ &= (x^2 - 3x + 1)^{(m-3)}(x - 1)^{(n-2m)}F_1(x) \end{aligned} \quad (1)$$

where $F_1(x) = -(n - m - 3)x^4 + (3n - 3m - 11)x^3 - (n - 13)x^2 - 4x$.

If $n - m \leq 2$ and $n \leq 12$, it is obvious that Eq.(1) is a polynomial on x with order $n - 2$, and each factor of Eq.(1) is a real polynomial with alternate coefficients on positive and negative, assume $\text{Eq.}(1) = \sum_{i=2}^{n-1} (-1)^i a_i x^{n-i}$, where $a_i > 0$ for $i = 2, 3, \dots, n - 1$. Notice that $a_i = \varphi_i(G_3^1) - \varphi_i(G_3^2) > 0$, thus $\varphi_i(G_3^1) > \varphi_i(G_3^2)$ in this case. If $n = 6, m = 3$, then $\text{Eq.}(1) = -2x^3 + 7x^2 - 4x = \sum_{i=3}^5 (-1)^i a_i x^{6-i}$, where $a_i > 0$. Since $a_i = \varphi_i(G_3^1) - \varphi_i(G_3^2)$, thus $\varphi_i(G_3^1) > \varphi_i(G_3^2)$ in this case. For other case, it is easy to find that all the signless Laplacian coefficients of G_3^1 and G_3^2 are not comparable.

And,

$$\begin{aligned} & Q(G_4^1, x) - Q(G_4^2, x) \\ &= (x^2 - 3x + 1)^{(m-3)}(x - 1)^{(n-2m-1)}x F_2(x) \end{aligned} \quad (2)$$

where $F_2(x) = x^4 - (n - m + 4)x^3 + (3n - 3m + 6)x^2 - (n + 3)x$.

By the fact $n > m$, in a similar way to the above discussion, it is easy to see $\varphi_i(G_4^1) \geq \varphi_i(G_4^2)$, and for $i = 2, 3, \dots, n - 1$, the inequality is strict.

And,

$$\begin{aligned} & Q(G_4^3, x) - Q(G_4^4, x) \\ &= (x^2 - 3x + 1)^{(m-5)}(x - 1)^{(n-2m-1)}x(x^2 - 4x + 2)F_3(x) \end{aligned} \quad (3)$$

where $F_3(x) = x^6 - (n - m + 6)x^5 + (5n - 5m + 16)x^4 - (7n - 6m + 25)x^3 + (2n + 22)x^2 - 8x$.

Similarly, $\varphi_i(G_4^3) \geq \varphi_i(G_4^4)$ holds, and for $i = 2, 3, \dots, n - 1$, the inequality is strict.

Furthermore,

$$\begin{aligned} & Q(G_4^4, x) - Q(G_4^2, x) \\ &= (x^2 - 3x + 1)^{(m-5)}(x - 1)^{(n-2m)}xF_4(x) \end{aligned} \quad (4)$$

where $F_4(x) = (n - m - 4)x^7 - (10n - 10m - 42)x^6 + (37n - 36m - 170)x^5 - (63n - 56m - 338)x^4 + (51n - 36m - 345)x^3 - (20n - 9m - 171)x^2 + (3n - 33)x$.

When $n = 8$, by using Matlab 7.0, $\varphi_i(G_4^4) \geq \varphi_i(G_4^2)$ holds. When $n = 9, 10$, by using Matlab 7.0, we can find that all the signless Laplacian coefficients of G_4^2 and G_4^4 are not comparable. If $n \geq 11$, and at least one of the following statements holds:

- (1). $m \geq 4, n - m \geq 7$.
- (2). $m \geq 5, n - m \geq 6$.
- (3). $m \geq 7, n - m \geq 5$.

In a similar way to the above discussion, it is easy to see $\varphi_i(G_4^4) \geq \varphi_i(G_4^2)$, and for $i = 2, 3, \dots, n - 1$, the inequality is strict. For other case, we claim that all the signless Laplacian coefficients of G_4^2 and G_4^4 are not comparable.

Remark. For a fixed value n , and for an arbitrary graph $H_1 \in \mathcal{G}_3(n)$ and an arbitrary graph $H_2 \in \mathcal{G}_4(n)$, all the signless Laplacian coefficients of H_1 and H_2 are not comparable. Since $\varphi_n(H_1) = 4, \varphi_n(H_2) = 0$, but $\varphi_{n-1}(H_1) = 3n, \varphi_{n-1}(H_2) = 4n$.

For fixed values n and m , and for arbitrary graphs $U_1 \in \mathcal{G}_3(n, m), U_2 \in \mathcal{G}_4(n, m)$, it is easy to see that $\varphi_{n-1}(U_1) < \varphi_{n-1}(U_2), \varphi_n(U_1) > \varphi_n(U_2)$, then H_1 and H_2 are not comparable with regard to all the signless Laplacian coefficients.

4 The Signless Laplacian coefficients of unicyclic graphs in $\mathcal{G}(n, m)$

Theorem 4.1 *In the set of all n -vertex unicyclic graphs in $\mathcal{G}_{g_1}(n, m)$, then*

- (1). *If $m = 2$, then G_3^1 has minimal all the signless Laplacian coefficients.*
- (2). *If $n - m \leq 3$ and $6 \leq n \leq 12, m \geq 3$, G_3^2 has minimal all the signless Laplacian coefficients $\varphi_i, i = 0, 1, 2, \dots, n$.*
- (3). *If $n - m > 3, m \geq 3$ or $n > 12, m \geq 3$, G_3^1 and G_3^2 are two extremal graphs which have minimal signless Laplacian coefficients $\varphi_i, i = 0, 1, 2, \dots, n$ in $\mathcal{G}_{g_1}(n, m)$. Furthermore, G_3^1 and G_3^2 can not be compared.*

Proof. For $n = 5$, the matching numbers of all unicyclic graphs in $\mathcal{G}_{g_1}(n, m)$ are 2, so by using Matlab 7.0, we have G_3^1 has minimum all the signless Laplacian coefficients in $\mathcal{G}_{g_1}(5, m)$. Next assume $n \geq 6$.

For fixed values n, m , let G be an arbitrary graph in $\mathcal{G}_{g_1}(n, m)$. Let $M(G)$ denote a maximum matching of G containing the most pendent edges. Now we need to prove after series of transformations, G will become to U' and U' has minimum signless Laplacian coefficients in $\mathcal{G}_{g_1}(n, m)$.

Step 1: When there is a pendent path $u_1 u_2 \dots u_k$ of length at least 3 in G , where $d(u_1) \geq 3, d(u_2) = d(u_3) = \dots = d(u_{k-1}) = 2, d(u_k) = 1$ and $k \geq 3$. Assume $u_{k-3} u_{k-2} \in M(G)$, since $E_{u_{k-3} u_{k-2}}^{u_k-2} \cap M(G) = \emptyset$, after performing the transformation of Definition 2.1 to $u_{k-3} u_{k-2}$, we have $M(G) = M(G_{u_{k-3} u_{k-2}})$, thus $G_{u_{k-3} u_{k-2}} \in \mathcal{G}_{g_1}(n, m)$, and $\varphi_i(G) > \varphi_i(G_{u_{k-3} u_{k-2}}), i = 2, 3, \dots, n-1$ by Theorem 2.2.

After performing transformations in Step 1 consecutively, we have that the graph in which each pendent path has length at most 2 has smaller signless Laplacian coefficients in $\mathcal{G}_{g_1}(n, m)$.

Step 2: For $u \in \{u : \text{dist}(u, C) \geq \text{dist}(u', C), d(u), d(u') \geq 3\}$, v is a neighbor of u , which satisfies $\text{dist}(u, C) - 1 = \text{dist}(v, C)$.

Case 2.1: When there is a pendent edge uu' at u , then after performing the transformation of Definition 2.3 to uv , we have $M(G'_{uv}) = m$ or $m + 1$. If $M(G'_{uv}) = m + 1$, then by the transformation of Definition 2.1 to ww' , we get a connected unicyclic graph U with $M(U) = m$, then by Theorem 2.2 and Theorem 2.4, $\varphi_i(G) > \varphi_i(G'_{uv}) > \varphi_i(U), i = 2, 3, \dots, n-1$. If $M(G_{uv}) = m$, then by Theorem 2.2, $\varphi_i(G) > \varphi_i(G'_{uv}), i = 2, 3, \dots, n-1$.

Case 2.2: When all pendent paths at u have lengths 2, then after performing

the transformation of Definition 2.1 to uv , by the fact $E_{uv}^u \cap M(G) = \emptyset$, we have $M(G) = M(G_{uv})$, and $\varphi_i(G) > \varphi_i(G_{uv}), i = 2, 3, \dots, n-1$ by Theorem 2.2.

After performing Step 2, the distance between the furthest branch vertex from the cycle C and C will be decreased. By performing transformations in Step 2 consecutively, it turns out that $G_{g_1}(s_1, t_1; s_2, t_2, \dots, s_{g_1}, t_{g_1})$ has smaller signless Laplacian coefficients in $\mathcal{G}_3(n, m)$.

Step 3: When $g(G) \geq 5$, we distinguish this case into the following two cases.

Case 3.1: If the edge $u_i u_{i+1}$ on the cycle belongs to $M(G)$, then by the assumption of $M(G)$, $t_i = t_{i+1} = 0$ holds, by performing the transformation of Definition 2.5 to u_i, u_{i+1}, u_{i+2} , we have $M(G) = M(G')$ and $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n-1$ by Theorem 2.6.

Case 3.2: If $t_i = t_{i+1} = 0$, and $u_i u_{i+1} \notin M(G)$. This case will not occur, since $M(G) \cup u_i u_{i+1}$ is also a matching of G , a contradiction to the assumption of $M(G)$.

Case 3.3: Assume $t_{i+1} \neq 0$, $u_{i+1} u'_{i+1}$ is a pendent edge at u_{i+1} and $u_{i+1} u'_{i+1} \in M(G)$. We distinguish this case into the following three cases.

Assume that $t_i \neq 0$, when $t_{i+2} = 0$, by performing the transformation of Definition 2.5 to u_i, u_{i+1}, u_{i+2} , we get G' with $M(G) = M(G')$, and $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n-1$ by Theorem 2.6. When $t_{i+2} \neq 0$, after performing the transformation of Definition 2.7 to u_i, u_{i+1}, u_{i+2} , we get G' with $M(G) = M(G')$, and $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n-1$ by Theorem 2.8.

Assume $t_i = 0, s_i \neq 0$, we perform the transformation of Definition 2.1 to the edge $u_i u_{i1}$ which is incident to u_i and belongs to the pendent path of length 2 at u_i . Then by the fact $E_{u_i u_{i1}}^{u_i} \cap M(G) = \emptyset$, we have $M(G) = M(G_{u_i u_{i1}})$, and $\varphi_i(G) > \varphi_i(G_{u_i u_{i1}}), i = 2, 3, \dots, n-1$ by Theorem 2.2.

Assume $t_i = s_i = 0$, when $t_{i+2} \neq 0$, by performing the transformation of Definition 2.5 to u_i, u_{i+1}, u_{i+2} , we have $M(G) = M(G')$ and $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n-1$ by Theorem 2.6. When $t_{i+2} = 0$, by performing the transformation of Definition 2.5 to u_i, u_{i+1}, u_{i+2} , we have $M(G) = M(G') + 1$, then by performing the transformation of Definition 2.1 to $u_i u_{i+1}$ of G' , we can get a connected unicyclic graph W with $M(W) = M(G)$, and $\varphi_i(G) > \varphi_i(G') > \varphi_i(W), i = 2, 3, \dots, n-1$ by Theorem 2.6 and Theorem 2.2.

Therefore, after taking Step 3, we have $U' \in \mathcal{G}_3(n, m)$. Then by Lemma 3.2, Lemma 3.3, and the discussion of Section 3, thus if $m = 2$, $U' \cong G_3^1$. If $n - m \leq 3$ and $6 \leq n \leq 12, m \geq 3$, $U' \cong G_3^2$. If $n - m > 3, m \geq 3$ or $n > 12, m \geq 3$, $U' \cong G_3^1$ or $U' \cong G_3^2$, and G_3^1, G_3^2 are not comparable with respect to all the signless Laplacian

coefficients. ■

Similar to the proof of Theorem 4.1, we have the following theorem.

Theorem 4.2 *In the set of all n -vertex unicyclic graphs in $\mathcal{G}_{g_2}(n, m)$, then*

- (1). *If $m = 2$, then G_4^1 has minimal all the signless Laplacian coefficients.*
- (2). *If $m = 3$, then G_4^2 has minimal all the signless Laplacian coefficients.*
- (3). *If $m \geq 4$, and $n = 8$, G_4^4 has minimal all the signless Laplacian coefficients $\varphi_i, i = 0, 1, 2, \dots, n$.*
- (4). *If $m \geq 4, n \geq 11$, and $n - m \geq 7$, or $m \geq 5, n \geq 11$, and $n - m \geq 6$, or $m \geq 7, n \geq 11$, and $n - m \geq 5$, then G_4^2 has minimal all the signless Laplacian coefficients $\varphi_i, i = 0, 1, 2, \dots, n$.*
- (5). *For other cases which n, m satisfy, G_4^2 and G_4^4 are two extremal graphs which have minimal signless Laplacian coefficients $\varphi_i, i = 0, 1, 2, \dots, n$ in $\mathcal{G}_{g_2}(n, m)$. Furthermore, G_4^2 and G_4^4 can not be compared.*

The proof is left to the reader.

Remark. From the discussion in Section 3, the extremal graphs with respect to all the signless Laplacian coefficients in $\mathcal{G}_{g_1}(n, m)$ and $\mathcal{G}_{g_2}(n, m)$ can not be compared.

By Theorem 1.3 and Theorem 4.1, Theorem 4.2, we obtained the following two corollaries.

Corollary 4.3 *If $m = 2$, then for $G \in \mathcal{G}_{g_1}(n, m)$, we have $IE(G) \geq IE(G_3^1)$. with equality if and only if $G \cong G_3^1$. If $m \geq 3$, then for $G \in \mathcal{G}_{g_1}(n, m)$, we have $IE(G) \geq \min\{IE(G_3^1), IE(G_3^2)\}$, with equality only if $G \cong G_3^1$ or $G \cong G_3^2$.*

Corollary 4.4 *If $m = 2$, then for $G \in \mathcal{G}_{g_2}(n, m)$, we have $IE(G) \geq IE(G_4^1)$. with equality if and only if $G \cong G_4^1$. If $m = 3$, then for $G \in \mathcal{G}_{g_2}(n, m)$, we have $IE(G) \geq IE(G_4^2)$. with equality if and only if $G \cong G_4^2$. If $m \geq 4$, then for $G \in \mathcal{G}_{g_2}(n, m)$, we have $IE(G) \geq \min\{IE(G_4^2), IE(G_4^4)\}$, with equality only if $G \cong G_4^2$ or $G \cong G_4^4$.*

5 The Signless Laplacian coefficients of unicyclic graphs in $\mathcal{G}(n)$

For convenience, denote $G_3(0, 0; 0, 0; 0, n-3)$ as S'_3 , and denote $G_4(0, 0; 0, 0; 0, 0; 0, n-4)$ as S'_4 .

Theorem 5.1 *In the set of all n -vertex unicyclic graphs in $\mathcal{G}_{g_1}(n)$, S'_3 has minimum signless Laplacian coefficients $\varphi_i, i = 0, 1, 2, \dots, n-1, n$.*

Proof. For $n = 5$, S'_3 minimize all the signless Laplacian coefficients. This result can be obtained by using Matlab 7.0.

For $n \geq 6$, let G be an arbitrary graph in $\mathcal{G}_{g_1}(n)$. We need to prove after series of transformations, G will become to S'_3 and S'_3 has minimum signless Laplacian coefficients in $\mathcal{G}_{g_1}(n)$.

Step 1: When there is a non-pendent edge uv which is not on the cycle. By performing the transformation of Definition 2.1 to uv , we have $G_{uv} \in \mathcal{G}_{g_1}(n)$, and $\varphi_i(G) > \varphi_i(G_{uv}), i = 2, 3, \dots, n-1$ by Theorem 2.2.

After performing Step 1 consecutively, we have that $G_{g_1}(0, t_1; 0, t_2; \dots; 0, t_{g_1})$ has smaller signless Laplacian coefficients in $\mathcal{G}_{g_1}(n)$.

Step 2: When $g(G) \geq 5$, by performing the transformation of Definition 2.5 to u_i, u_{i+1}, u_{i+2} , we have $\varphi_i(G) > \varphi_i(G'), i = 2, 3, \dots, n-1$ by Theorem 2.6. Then by performing the transformation of Definition 2.1 to the new pendent edge $u_{i+1}u_{i+2}$, a new graph with smaller signless Laplacian coefficients is obtained by Theorem 2.2.

Therefore, after taking Step 2 consecutively, we obtain the extremal graph with minimal signless Laplacian coefficients which must belong to $\mathcal{G}_3(n)$. Then by Lemma 3.1, Lemma 3.4, and the discussion of Section 3, thus S'_3 is the resulting graph which has minimum signless Laplacian coefficients in $\mathcal{G}_{g_1}(n)$. ■

Theorem 5.2 *In the set of all n -vertex unicyclic graphs in $\mathcal{G}_{g_2}(n)$, S'_4 has minimum signless Laplacian coefficients $\varphi_i, i = 0, 1, 2, \dots, n-1, n$.*

Proof. For $n = 5$, S'_4 is the unique graph $\mathcal{G}_{g_2}(5)$, so it has the minimum signless Laplacian coefficients in $\mathcal{G}_{g_2}(5)$.

For $n \geq 6$, let G be an arbitrary graph in $\mathcal{G}_{g_2}(n)$. The method in the proof of Theorem 5.1 can be used to prove this result similarly. ■

Remark. From the discussion in Section 3, the extremal graphs S'_3 and S'_4 with respect to all the signless Laplacian coefficients in $\mathcal{G}_{g_1}(n)$ and $\mathcal{G}_{g_2}(n)$ can not be compared.

By Theorem 1.3 and Theorem 5.1, Theorem 5.2, we obtained the following two corollaries.

Corollary 5.3 *In the set of all n -vertex unicyclic graphs in $\mathcal{G}_{g_1}(n)$, S'_3 is the unique graph with the minimal IE .*

Corollary 5.4 *In the set of all n -vertex unicyclic graphs in $\mathcal{G}_{g_2}(n)$, S'_4 is the unique graph with the minimal IE .*

References

- [1] D. Cvetković, P. Rowlinson, S. Simić, *Signless Laplacians of finite graphs*, Linear Algebra Appl. 432 (2007) 155-171.
- [2] D. Cvetković, P. Rowlinson, S. Simić, *Eigenvalue bounds for the signless Laplacian*, Publ. Inst. Math. (Beograd) 81 (95) (2007) 11-27.
- [3] D. Cvetković, S. Simić, *Towards a spectral theory of graphs based on the signless Laplacian, I*, Publ. Inst. Math. (Beograd) 85 (99) (2009) 19-33.
- [4] D. Cvetković, S. Simić, *Towards a spectral theory of graphs based on the signless Laplacian, II*, Linear Algebra Appl. 432 (2010) 2257-2272.
- [5] D. Cvetković, S. Simić, *Towards a spectral theory of graphs based on the signless Laplacian, III*, Appl. Anal. Discrete Math. 4 (2010) 156-166.
- [6] J. M. Guo, *On the second largest Laplacian eigenvalue of trees*, Linear Algebra Appl. 404 (2005) 251-261.
- [7] I. Gutman, D. Kiani, M. Mirzakhah, *On incidence energy of graphs*, Match 62 (2009) 573-580.
- [8] I. Gutman, D. Kiani, M. Mirzakhah, B. Zhou, *On incidence energy of a graph*, Linear Algebra Appl. 431 (2009) 1223-1233.
- [9] C. X. He, J. Y. Shao, J. L. He, *On the Laplacian spectral radii of bicyclic graphs*, Discrete Math. 308 (2008) 5981-5995.
- [10] C. X. He, H. Y. Shan, *On the Laplacian coefficients of bicyclic graphs*, Discrete Math. 310 (2010) 3404-3412.
- [11] S. S. He, S. C. Li, *Ordering of trees with fixed matching number by the Laplacian coefficients*, Linear Algebra Appl. 435 (2011) 1171-1186.

- [12] A. Ilić, M. Ilić, *Laplacian coefficients of trees with given number of leaves or vertices of degree two*, Linear Algebra Appl. 431 (2009) 2195-2202.
- [13] A. Ilić, *On the ordering of trees by the Laplacian coefficients*, Linear Algebra Appl. 431 (2009) 2203-2212.
- [14] A. Ilić, *Trees with minimal Laplacian coefficients*, Comput. Math. Appl. 59 (2010) 2776-2783.
- [15] M. Jooyandeh, D. Kiani, M. Mirzakhah, *Incidence energy of a graph*, Match 62 (2009) 561-572.
- [16] A. K. Kelmans, V. M. Chelnokov, *A certain polynomial of a graph and graphs with extremal number of trees*, J. Combin. Theory, Ser. B 16 (1974) 197-214.
- [17] B. Mohar, *On the Laplacian coefficients of acyclic graphs*, Linear Algebra Appl. 722 (2007) 736-741.
- [18] M. Mirzakhah, D. Kiani, *Some results on signless Laplacian coefficients of graphs*, Linear Algebra Appl. 437 (2012) 2243-2251.
- [19] D. Stevanović, A. Ilić, *On the Laplacian coefficients of unicyclic graphs*, Linear Algebra Appl. 430 (2009) 2290-2300.
- [20] S. W. Tan, *On the Laplacian coefficients of unicyclic graphs with prescribed matching number*, Discrete Math. 311 (2011) 582-594.
- [21] X. D. Zhang, X. P. Lv, Y. H. Chen, *Order trees by the Laplacian coefficients*, Linear Algebra Appl. 431 (2009) 2414-2424.